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Pricing and hedging short-maturity Asian options in local volatility models

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Abstract

This paper discusses the short-maturity behavior of Asian option prices and hedging portfolios. We consider the risk-neutral valuation and the delta value of the Asian option having a Hölder continuous payoff function in a local volatility model. The main idea of this analysis is that the local volatility model can be approximated by a Gaussian process at short maturity T. By combining this approximation argument with Malliavin calculus, we conclude that the short-maturity behaviors of Asian option prices and the delta values are approximately expressed as those of their European counterparts with volatility

$$\sigma_A(T) := \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t, S_0)(T-t)^2\, dt}\,,$$

where $\sigma(\cdot, \cdot)$ is the local volatility function and S_0 is the initial value of the stock. In addition, we show that the convergence rate of the approximation is determined by the Hölder exponent of the payoff function. Finally, the short-maturity asymptotics of Asian call and put options are discussed from the viewpoint of the large deviation principle.

Key words: Asian option, short maturity, Hölder continuous, local volatility model, Gaussian process, Malliavin calculus, large deviation principle
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Chapter 1

Introduction

This paper focuses on an *arithmetic average Asian option* in continuous time having a terminal payoff of the form

$$\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right) .$$

Here, the function $\Phi : \mathbb{R} \to \mathbb{R}$ is a prescribed payoff function, T is a constant that denotes the maturity, and $(S_t)_{t\geq 0}$ is an underlying price process. For conciseness, we refer to this option as the *Asian option*. Because of its average property, the Asian option is less exposed to a sudden plummet in stock prices just before maturity. In particular, for hedging purposes, the Asian option is attractive to many traders and financial institutions. For an overview of the role of the Asian option in the financial market, see Wilmott (2006).

Despite its popularity in the real market, the Asian option is mathematically challenging to price and hedge in general. Even when the underlying stock price $(S_t)_{t\geq 0}$ follows the classical Black–Scholes model, no simple closed-form formula for the density of the random variable $\frac{1}{T}\int_0^T S_t\,dt$ is known. In this paper, we analyze the Asian option for pricing and hedging purposes in the short-maturity regime.

We focus on the case in which the payoff function Φ is any Hölder continuous function and the process $(S_t)_{t\geq 0}$ follows a local volatility model. Detailed assumptions on the model are presented in Chapter 2. This paper primarily deals with two features of the Asian option. The first feature is the short-maturity behavior of the option price. The short-maturity Asian option price is shown to be determined by the *Asian volatility*, which is defined by

$$\sigma_A(T) := \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t,S_0)(T-t)^2\,dt}\,,$$

where $\sigma(\cdot,\cdot)$ is a local volatility function. The second feature is the initial-value sensitivity of the Asian option. This type of sensitivity is widely referred to as *delta* in the finance literature. In the modern theory of finance, the delta value is used to hedge financial derivatives. This paper shows that the delta value can be expressed in terms of the Asian volatility for small T, as with the option price. In summary, the Asian option price $P_A(T)$ and delta value $\Delta_A(T)$ are expressed as

$$\begin{aligned} P_A(T) &= \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T^\gamma)\,, \\ \Delta_A(T) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,, \end{aligned}$$

for a standard normal random variable Z and the Hölder exponent γ of the payoff function Φ . The asymptotic estimates established in this paper are particularly meaningful with regard to the non-linear payoff function Φ . For example, let us consider a payoff function Φ that equals $x \mapsto (x-K)_+^\gamma$ in some neighborhood of K for some $\frac{1}{2} < \gamma < 2$. If K equals the initial value S_0 , we prove that the leading order of delta is $T^{\frac{\gamma-1}{2}}$ and we provide its coefficient in a rigorous manner.

As a special case, estimates for the Asian call and put option delta values are enhanced. This paper supplements the asymptotic result in Pirjol and Zhu (2016) in two ways. First, we prove that the rate function of the out-of-the-money (OTM) Asian option delta value is the same as that of the OTM Asian option price. Second, a precise Taylor expansion of the in-the-money (ITM) Asian option delta value is provided.

Estimates for the price and delta of the *European option* having the terminal payoff $\Phi(S_T)$ are also investigated. Short-maturity formulas for the European option prices and delta values are obtained if the Asian volatility is replaced by

$$\sigma_E(T) := \sqrt{\frac{1}{T}\int_0^T \sigma^2(t, S_0)\,dt}\,,$$

which we refer to as the *European volatility*, in the formulas for the Asian option prices and delta values. With regard to $\sigma_A(T)$ and $\sigma_E(T)$, we compare the Asian option with the European option in Chapter 6. In addition to the European option, the *geometric average Asian option* having the terminal payoff

$$\Phi\left(e^{\frac{1}{T}\int_0^T \log S_t\,dt}\right)$$

is also compared in the Black–Scholes model.

To obtain these estimates, we incorporate many well-known mathematical techniques with the approximation scheme. The main technique is L^p-approximation of the underlying stock price $(S_t)_{0\le t\le T}$ by some Gaussian process $(\hat{X}_t)_{0\le t\le T}$. Precise arguments are presented in Chapter 2. We adopt the method used in Pirjol and Zhu (2016, 2019); Pirjol et al. (2019), where the same idea was used to compute the short-maturity asymptotics of at-the-money(ATM) Asian call and put option prices. On the basis of this idea, our research focus shifts from the random variable $\frac{1}{T}\int_0^T S_t\,dt$ having a sophisticated density to the Gaussian random variable $\frac{1}{T}\int_0^T \hat{X}_t\,dt$. This is the key strategy that we adopt to approximate the Asian option throughout in Chapters 2–4. In addition, we use Malliavin calculus theory to analyze the Asian option delta value. In Benhamou (2000); Pirjol and Zhu (2018), the authors used Malliavin calculus for their sensitivity analysis of the Asian call and put option. We use their methods to express the Asian option delta value. Furthermore, we use the large deviation principle to examine both OTM and ITM Asian call and put options. The large deviation principle was first used to investigate the short-maturity Asian option in Pirjol and Zhu (2016, 2019).

Our study is of practical interest because existing numerical methods have proven to be less efficient in the case of short maturity or low volatility. Numerical analysis of the Asian option was conducted in Geman and Yor (1993); Linetsky (2004); Broadie et al. (1999); Boyle and Potapchik (2008). However, as pointed out in Fu et al. (1999); Vecer (2002), such methods are either problematic in the short-maturity regime or computationally expensive. We expect our analysis to help overcome the numerical inefficiency in the short-maturity regime.

Recently, the short-maturity Asian option has been studied by many researchers. Under a local volatility model, the asymptotics of Asian option price have been investigated in Pirjol and Zhu (2016); Pirjol et al. (2019). In Pirjol and Zhu (2019), asymptotic analysis was conducted under the constant elasticity of the variance model. The above-mentioned studies have used the large deviation principle. They have analytically solved the rate function of the law of $\frac{1}{T}\int_0^T S_t\,dt$ for approximation. Sensitivity analysis was conducted in Pirjol and Zhu (2018) as a follow-up study under the Black–Scholes model. On the basis of the approximated option price established in Pirjol and Zhu (2016), the sensitivities have been examined in Pirjol and Zhu (2018).

Compared to the above-mentioned studies, the contributions of our study are threefold. First, our paper focuses on a model having a time-dependent diffusion term. The analysis performed in Pirjol and Zhu (2016); Pirjol et al. (2019) was based on the assumption that the diffusion is time-independent. The obtained rate function was strongly dependent on this time-independent assumption. Second, we provide the leading order and its exact coefficient for an arbitrary Hölder continuous payoff function Φ. This generalizes the results in Pirjol and Zhu (2016, 2019), where vanilla options(call and put) were mainly considered. Finally, in contrast to Pirjol and Zhu (2018), our estimates for delta do not build upon the approximated option price. Thus, our estimates are free from controlling nested errors.

Malliavin calculus theory has long been applied to the Asian option. The necessary and sufficient conditions on the Malliavin weights of sensitivities have been studied in Benhamou (2000). The Malliavin weights under the Black–Scholes model have been computed in Boyle and Potapchik (2008); Grasselli and Hurd (2005). Relevant theories have been presented in Nualart (1995); Nualart and Nualart (2018). Furthermore, under a general diffusion process, the prices of continuously sampled Asian options have been estimated using Malliavin calculus in Gobet and Miri (2014). In Shiraya et al. (2011), price bounds for discretely sampled options have been presented for a stochastic volatility model.

The remainder of this paper is organized as follows. Chapter 2 outlines the model setup and introduces six auxiliary processes that are used to approximate $(S_t)_{t\geq 0}$ in the $L^p(\mathbb{Q})$ norm. Chapter 3 examines the Asian option price for small T when the payoff Φ is Lipschitz continuous. Under the same assumption on Φ , Chapter 4 investigates the Asian option delta value. Chapter 5 generalizes the results from Chapters 3 and 4 to the Hölder continuous payoff Φ . Chapter 6 concatenates the asymptotic results from Chapters 3 and 4, and compares them with their European counterparts. Chapter 7 uses the large deviation principle to study the Asian call and put option. Finally, Chapter 8 concludes the paper.

Chapter 2

Approximation scheme

We analyze the short-maturity asymptotic behavior of Asian options under local volatility models. Assume that the stock price process $(S_t)_{t\geq 0}$ follows a local volatility model,

$$dS_t = (r-q)S_t\,dt + \sigma(t,S_t)S_t\,dW_t\,, \quad S_0 > 0\,, \tag{2.0.1}$$

under risk-neutral measure $\mathbb{Q}$, where r is the short rate, q is the dividend rate, and $(W_t)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion. From Assumption 1 below, there exists a unique strong solution for this stochastic differential equation.

Assumption 1. *Let us consider the following assumptions for the diffusion function.*

(i) The function $\sigma(t,x)$ is measurable in $[0,\infty)\times\mathbb{R}$ and is bounded, i.e., there are two constants $\underline{\sigma}$ and $\overline{\sigma}$ such that $0<\underline{\sigma}\leq\sigma(t,x)\leq\overline{\sigma}<\infty$ for all t and x.

(ii) For each t, the function $\sigma(t,\cdot)$ is twice differentiable in $\mathbb{R}$.

(iii) Define $\nu(t,x) := \frac{\partial}{\partial x}\sigma(t,x)x$ and $\rho(t,x) := \frac{\partial^2}{\partial x^2}\sigma(t,x)x$. Then, for each t, functions $\sigma(t,\cdot), \sigma(t,\cdot)\cdot, \nu(t,\cdot), \rho(t,\cdot)$ are Lipschitz continuous with a Lipschitz coefficient $\alpha>0$. More precisely, there is a constant

$\alpha > 0$ *such that for any* $x, y \in \mathbb{R}$,

$$\sup_{t\geq 0} |\sigma(t,x) - \sigma(t,y)| \leq \alpha\,|x-y|\,,\ \sup_{t\geq 0} |\sigma(t,x)x - \sigma(t,y)y| \leq \alpha\,|x-y|\,,$$

$$\sup_{t\geq 0} |\nu(t,x) - \nu(t,y)| \leq \alpha\,|x-y|\,,\ \sup_{t\geq 0} |\rho(t,x) - \rho(t,y)| \leq \alpha\,|x-y|\ .$$

Clearly, this assumption covers the Black–Scholes model. In this paper, only Hölder continuous payoff Φ will be considered. Under Assumption 2, β and γ always refer to constants with regard to Φ throughout this paper.

Assumption 2. *The payoff function* $\Phi : \mathbb{R} \to \mathbb{R}$ *is* γ*-Hölder continuous with coefficient* $\beta > 0$*. More precisely, for any* $x, y \in \mathbb{R}$*,* $|\Phi(x) - \Phi(y)| \leq \beta|x-y|^\gamma$ *with* $0 < \gamma \leq 1$.

To clarify the arguments, we will first consider Lipschitz continuous payoff Φ in Chapters 3 and 4. Unless stated otherwise, β always refers to the Lipschitz coefficient.

Assumption 3. *The payoff function* $\Phi : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous with coefficient* $\beta > 0$*. More precisely, for any* $x, y \in \mathbb{R}$*,* $|\Phi(x) - \Phi(y)| \leq \beta|x-y|$.

Now, we introduce six processes that are used to approximate $(S_t)_{t\geq 0}$ in $L^p(\mathbb{Q})$:

$$X,\, Y,\, \tilde{X},\, \tilde{Y},\, \hat{X},\, \hat{Y}\,.$$

Define a process $(X_t)_{t\geq 0}$ as

$$dX_t = \sigma(t, X_t)X_t\, dW_t\,, \quad X_0 = S_0 > 0 \tag{2.0.2}$$

and its first variation process as

$$dY_t = \nu(t, X_t)Y_t\, dW_t\,, \quad Y_0 = 1\,. \tag{2.0.3}$$

These two processes will be used to approximate the underlying process $(S_t)_{t\geq 0}$ in Chapters 3 and 4. We also define two geometric Gaussian processes $(\tilde{X}_t)_{t\geq 0}$ and $(\tilde{Y}_t)_{t\geq 0}$ as

$$\begin{aligned} d\tilde{X}_t &= \sigma(t, S_0)\tilde{X}_t \, dW_t \,, & \tilde{X}_0 &= S_0 \,, \\ d\tilde{Y}_t &= \nu(t, S_0)\tilde{Y}_t \, dW_t \,, & \tilde{Y}_0 &= 1 \,. \end{aligned} \tag{2.0.4}$$

In Lemma 2.0.1, these two processes will be used to approximate $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ in the $L^p(\mathbb{Q})$ norm at short time. Finally, we define two Gaussian processes $(\hat{X}_t)_{t\geq 0}$, $(\hat{Y}_t)_{t\geq 0}$ by

$$\begin{aligned} d\hat{X}_t &= \sigma(t, S_0)S_0 \, dW_t \,, & \hat{X}_0 &= S_0 \,, \\ d\hat{Y}_t &= \nu(t, S_0) \, dW_t \,, & \hat{Y}_0 &= 1 \,. \end{aligned}$$

Furthermore, in Lemma 2.0.1, these two processes will be used to approximate $(\tilde{X}_t)_{t\geq 0}$ and $(\tilde{Y}_t)_{t\geq 0}$ in the $L^p(\mathbb{Q})$ norm at short time. It is easy to check that the six above-mentioned processes are all continuous martingale processes adapted to the Brownian filtration $(\mathcal{F}_t^W)_{t\geq 0}$. Now, we introduce Lemma 2.0.1. See Appendix A.1 for the proof.

Lemma 2.0.1. *Under Assumption 1, for any $p > 0$, there exists a positive constant B_p depending only on p such that the following inequalities hold.*

(i) For $0 \leq t \leq 1$,

$$\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] \leq B_p t^p, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \leq B_p t^p. \tag{2.0.5}$$

(ii) For $0 \leq t \leq 1$,

$$\mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^p] \leq B_p t^p, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_t - \hat{Y}_t|^p] \leq B_p t^p. \tag{2.0.6}$$

We now present the short-time behavior of the four processes $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$, $(\tilde{Y}_t)_{t\geq 0}$ in the following lemma. All the moments of the four random variables $X_T, \tilde{X}_T, Y_T, \tilde{Y}_T$ and their integrals over $[0, T]$ converge to constants as $T \to 0$. We rephrase this argument as the following technical statement for later use. The proof is provided in Appendix A.2.

Lemma 2.0.2. *Under Assumption 1, consider processes* $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$, $(\tilde{Y}_t)_{t\geq 0}$ *stated in Eqs.*(2.0.2), (2.0.3), *and* (2.0.4). *The process* $(Z_t^{p_1,p_2,p_3,p_4})_{t\geq 0}$, *which is defined by* $Z_t^{p_1,p_2,p_3,p_4} := X_t^{p_1}\tilde{X}_t^{p_2}Y_t^{p_3}\tilde{Y}_t^{p_4}$ *for any* $p_i \in \mathbb{R}$, $i \in \{1,2,3,4\}$, *satisfies following two statements.*

(i) $\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq t\leq T} Z_t^{p_1,p_2,p_3,p_4}\Big] < \infty$ *for any* $T > 0$.
Furthermore, $\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq t\leq T} Z_t^{p_1,p_2,p_3,p_4}\Big] = S_0^{p_1+p_2}$.

(ii) Moreover, for any $q_j \in \mathbb{R}$, $j \in \{1,2,\cdots 8\}$,

$$\begin{aligned}
&\lim_{T\to 0}\mathbb{E}^{\mathbb{Q}}\Bigg[Z_T^{q_1,q_2,q_3,q_4}\left(\frac{1}{T}\int_0^T X_t\,dt\right)^{q_5}\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)^{q_6} \\
&\times\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^{q_7}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{q_8}\Bigg] = S_0^{q_1+q_2+q_5+q_6}.
\end{aligned} \tag{2.0.7}$$

Chapter 3

Short-maturity limit of an option price with Lipschitz continuous payoffs

Under the risk-neutral measure $\mathbb{Q}$, the arbitrage-free values of the Asian and European options are

$$P_A(T) := e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\right]\,, \quad P_E(T) := e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)]\,,$$

where T is the maturity. Throughout this chapter, we impose Assumption 3 on Φ . Our objective is to find an asymptotic formula for the Asian option price up to $\mathcal{O}(T)$; a formula for its European counterpart is also presented. More precisely, we will prove in Theorem 3.0.1 that $P_A(T)$ and $P_E(T)$ are asymptotically equal to

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + S_0\sqrt{\frac{1}{T^2}\int_0^T \sigma^2(t,S_0)(T-t)^2\,dt\,Z}\right)\right]$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + S_0\sqrt{\int_0^T \sigma^2(t, S_0)\,dt}\,Z\right)\right],$$

respectively, where Z is a standard normal random variable.

To achieve these results, we perform two approximation steps. First, in Lemma 3.0.1, the underlying $(S_t)_{0\le t\le T}$ is approximated by $(X_t)_{0\le t\le T}$. Second, in Theorem 3.0.1, we approximate the process $(X_t)_{0\le t\le T}$ sequentially by $(\tilde{X}_t)_{0\le t\le T}$ and $(\hat{X}_t)_{0\le t\le T}$ using Lemma 2.0.1. The proof of Lemma 3.0.1 is provided in Appendix B.1.

Lemma 3.0.1. *Under Assumptions 1 and 3, as $T \to 0$, we have*

(i) $P_A(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] + \mathcal{O}(T)$,

(ii) $P_E(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] + \mathcal{O}(T)$.

As can be seen in Eqs.(2.0.1) and (2.0.2), the processes S and X have the same diffusion terms; however, the drift term of X is zero. Thus, this lemma implies that while estimating the Asian and European option prices, the drift of the underlying stock becomes negligible at small $T > 0$. This is similar to Theorem 2 of Pirjol and Zhu (2016) and Theorem 5 of Pirjol and Zhu (2019). In Pirjol and Zhu (2016, 2019), the rate function that governs the short-maturity behavior of the Asian call and put option was shown to be independent of the drift term.

The main result of this chapter is the following theorem, which states asymptotic formulas for the Asian and European option prices.

Theorem 3.0.1. *Under Assumptions 1 and 3, as $T \to 0$, we have*

(i) $$P_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + S_0\sqrt{\frac{1}{T^2}\int_0^T \sigma^2(t, S_0)(T-t)^2\,dt}\,Z\right)\right] + \mathcal{O}(T),$$

(ii) $$P_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0 + S_0\sqrt{\int_0^T \sigma^2(t, S_0)\,dt}\,Z\right)\right] + \mathcal{O}(T),$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

Proof. The statement actually directly comes from Lemmas 2.0.1 and 3.0.1. Observe that

$$\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right]-\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right]\right|\\
&\le \frac{\beta}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|X_t-\tilde{X}_t|]\,dt=\frac{\beta B_1}{2}T\,,\\
&\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right]-\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right]\right|\\
&\le \frac{\beta}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t-\hat{X}_t|]\,dt=\frac{\beta B_1}{2}T
\end{aligned}$$

for the positive constant B_1 in Lemma 2.0.1 and β in Assumption 3. Thus, together with Lemma 3.0.1, we get

$$P_A(T)=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right]+\mathcal{O}(T)\,.$$

From a direct calculation using the Fubini theorem regarding a stochastic integral,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\right]&=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+\frac{S_0}{T}\int_0^T\int_0^t \sigma(s,S_0)\,dW_s\,dt\right)\right]\\
&=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+\frac{S_0}{T}\int_0^T \sigma(s,S_0)(T-s)\,dW_s\right)\right]\,.
\end{aligned}$$

Hence, we get the desired result for $P_A(T)$. Applying the same argument to $P_E(T)$,

$$P_E(T)=\mathbb{E}^{\mathbb{Q}}[\Phi(\hat{X}_T)]+\mathcal{O}(T).$$

Using

$$\mathbb{E}^{\mathbb{Q}}[\Phi(\hat{X}_T)]=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+S_0\int_0^T \sigma(t,S_0)\,dW_t\right)\right]\,,$$

we get the desired result. $\square$

Corollary 3.0.2. *Under Assumptions 1 and 3, the prices of both the Asian option and the European option share the same limit $\Phi(S_0)$ as $T \to 0$ with the convergence order $\mathcal{O}(\sqrt{T})$. More precisely,*

$$P_A(T) = \Phi(S_0) + \mathcal{O}(\sqrt{T}) , \quad P_E(T) = \Phi(S_0) + \mathcal{O}(\sqrt{T}) .$$

We introduce two notions of volatilities called the *Asian volatility* and the *European volatility*.

Definition 3.0.1. *We define the Asian volatility $\sigma_A(T)$ and the European volatility $\sigma_E(T)$ as*

$$\begin{aligned} \sigma_A(T) &:= \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t, S_0)(T-t)^2 \, dt} \, , \\ \sigma_E(T) &:= \sqrt{\frac{1}{T}\int_0^T \sigma^2(t, S_0) \, dt} \, . \end{aligned} \tag{3.0.1}$$

In terms of the Asian volatility and the European volatility, Theorem 3.0.1 can be rewritten as

$$P_A(T) = \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T)$$

and

$$P_E(T) = \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)] + \mathcal{O}(T) .$$

In Chapter 4, short-maturity asymptotic formulas for the delta values will be presented in terms of the Asian volatility and the European volatility.

Remark 3.0.1. *From the definition of $\sigma_A(T)$ and $\sigma_E(T)$, we can observe that they depend only on the first argument t of the volatility function $\sigma(t,x)$. For any $0 \le t \le T$, the local behavior of $x \mapsto \sigma(t,x)$ near $x = S_0$ does not affect $\sigma_A(T)$ and $\sigma_E(T)$.*

We compute asymptotic results for the call and put options as an example of Theorem 3.0.1. This generalizes the result in Theorem 6 of Pirjol and Zhu (2016) for the ATM case.

Example 3.0.1. *Let P_A^{call} and P_A^{put} be the Asian call and put prices with the strike K, i.e., the payoff functions are $\Phi(x) = (x-K)_+$, and $\Phi(x) = (K-x)_+$, respectively. Then,*

$$P_A^{\text{call}}(T) = \begin{cases} 0 + \mathcal{O}(T), & \text{if } S_0 < K\,, \\ \frac{S_0\sigma_A(T)}{\sqrt{2\pi}}\sqrt{T} + \mathcal{O}(T), & \text{if } S_0 = K\,, \\ S_0 - K + \mathcal{O}(T), & \text{if } S_0 > K\,, \end{cases}$$

$$P_A^{\text{put}}(T) = \begin{cases} K - S_0 + \mathcal{O}(T), & \text{if } S_0 < K\,, \\ \frac{S_0\sigma_A(T)}{\sqrt{2\pi}}\sqrt{T} + \mathcal{O}(T), & \text{if } S_0 = K\,, \\ 0 + \mathcal{O}(T), & \text{if } S_0 > K\,. \end{cases}$$

The prices of the European call and put option are obtained by replacing $\sigma_A(T)$ in the above-mentioned expressions with $\sigma_E(T)$.

Example 3.0.2. *Given any $K\,, \delta > 0$ and $0 \le \epsilon < 1\,,$ define the payoff function Φ by*

$$\Phi(x) = (x-K)^{1+\epsilon}\mathbb{1}_{\{K \le x < K+\delta\}} + \delta^{1+\epsilon}\mathbb{1}_{\{K+\delta \le x\}}\,.$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$P_A(T) = \frac{1}{2}(S_0\sigma_A(T))^{1+\epsilon}\, M(1+\epsilon)T^{\frac{1+\epsilon}{2}} + \mathcal{O}(T)\,,$$

where $M(1+\epsilon) := \mathbb{E}^{\mathbb{Q}}[|Z|^{1+\epsilon}]$ with a standard normal variable Z. If we replace $\sigma_A(T)$ with $\sigma_E(T)$, we get the asymptotic result for the European option price $P_E(T)$.

Chapter 4

Short-maturity estimates for an option delta value with Lipschitz continuous payoffs

In this chapter, we present the short-maturity asymptotic for the sensitivity of the option with respect to the initial value S_0. In many studies, this sensitivity is referred to as *delta*. We follow this convention to define the Asian delta value and the European delta value as

$$\Delta_A(T) := \frac{\partial}{\partial S_0} P_A(T)\,, \quad \Delta_E(T) := \frac{\partial}{\partial S_0} P_E(T)\,.$$

Throughout this chapter, only the Lipschitz continuous payoff Φ will be considered. Our main objective is to obtain the short-maturity asymptotic for $\Delta_A(T)$, $\Delta_E(T)$. These asymptotic results are given in Theorems 4.2.1 and 4.3.1. In summary, for small $T > 0$,

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}} Z\right] + \mathcal{O}(\sqrt{T})\,,$$

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})\,,$$

where Z denotes a standard normal random variable.

We derive the above-mentioned formulas as follows. In Lemma 4.0.1, we first approximate $(S_t)_{0\le t\le T}$ by $(X_t)_{0\le t\le T}$ in line with Lemma 3.0.1. See Appendix C.1 for the proof.

Lemma 4.0.1. *Under Assumptions 1 and 3, as $T \to 0$, we have*

(i) $\Delta_A(T) = e^{-rT}\dfrac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\dfrac{1}{T}\displaystyle\int_0^T X_t\,dt\right)\right] + \mathcal{O}(\sqrt{T})\,,$

(ii) $\Delta_E(T) = e^{-rT}\dfrac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] + \mathcal{O}(\sqrt{T})\,.$

Next, in Sections 4.1 and 4.2, we present a short-maturity asymptotic formula for the Asian delta value $\Delta_A(T)$. A formula for the European delta value $\Delta_E(T)$ is presented in Section 4.3.

4.1 Approximation for the Malliavin representation of the Asian delta value

By Malliavin calculus, the Asian delta value can be represented by the weighted sum of the payoffs. The computation under the Black–Scholes model has already been presented in Boyle and Potapchik (2008); Benhamou (2000). Under the local volatility model, we describe a possible representation in the following proposition.

Proposition 4.1.1. *For the process X stated in Eq.*(2.0.2) *under Assumption 1, we have*

$$\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta\left(\frac{2Y_\cdot^2}{\sigma(\cdot,X_\cdot)X_\cdot\int_0^T Y_t\,dt}\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta\left(\frac{2Y_\cdot^2}{\sigma(\cdot,X_\cdot)X_\cdot}\right)\frac{1}{\int_0^T Y_t\,dt}\right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T \frac{2{Y_s}^2}{\sigma(s,X_s)X_s} D_s\left(\frac{1}{\int_0^T Y_t\,dt}\right)ds\right],
\end{aligned}
$$

where $\delta(\cdot)$ is the Skorokhod integral in $[0,T]$ and $D_s(\cdot)$ is the Malliavin derivative.

Proof. See Nualart (1995); Benhamou (2000). □

With this proposition, Lemma 4.0.1 implies that for small $T > 0$, the Asian delta value asymptotically behaves as

$$
\begin{aligned}
\Delta_A(T) &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_sF)\,ds\right] + \mathcal{O}(\sqrt{T})\,.
\end{aligned} \tag{4.1.1}
$$

Here, we define a process $(u_s)_{0\le s\le T}$ and a random variable F by

$$
u_s := \frac{2{Y_s}^2}{\sigma(s,X_s)X_s}\,, \quad F := \frac{1}{\int_0^T Y_t\,dt}\,.
$$

To investigate $\Delta_A(T)$, we will approximate the two expectations on the right-hand side of Eq.(4.1.1). The approximation of the first expectation,

$$
\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right], \tag{4.1.2}
$$

is described in Proposition 4.1.2, and the approximation of the second expectation,

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_sF)\,ds\right], \tag{4.1.3}$$

is described in Proposition 4.1.3. To analyze Eq.(4.1.2), we introduce two processes $(\tilde{u}_s)_{0\le s\le T}$ and $(\hat{u}_s)_{0\le s\le T}$ and two random variables $\tilde{F}$ and $\hat{F}$, defined by

$$\tilde{u}_s := \frac{2\tilde{Y}_s^2}{\sigma(s,\tilde{X}_s)\tilde{X}_s}, \quad \hat{u}_s := \frac{2\hat{Y}_s^2}{\sigma(s,\hat{X}_s)\hat{X}_s}\mathbb{1}_{\{\hat{X}_s\ge\frac{S_0}{2}\}}$$

and

$$\tilde{F} := \frac{1}{\int_0^T\tilde{Y}_t\,dt}, \quad \hat{F} := \frac{1}{\int_0^T\hat{Y}_t\,dt}\mathbb{1}_{\{\frac{1}{T}\int_0^T\hat{Y}_t\,dt\ge\frac{1}{2}\}},$$

where $\mathbb{1}_A$ denotes the indicator function of set A. In Lemma 4.1.1, the process $(u_s)_{0\le s\le T}$ is approximated using $(\tilde{u}_s)_{0\le s\le T}$ and $(\hat{u}_s)_{0\le s\le T}$. As for the random variable F, we use $\tilde{F}$ and $\hat{F}$. This procedure is similar to the approximation based on Lemma 2.0.1. The proof of Lemma 4.1.1 is given in Appendix C.2.

Remark 4.1.1. *The reason for introducing the indicator functions $\mathbb{1}_{\{\hat{X}_s\ge\frac{S_0}{2}\}}$ and $\mathbb{1}_{\{\frac{1}{T}\int_0^T\hat{Y}_t\,dt\ge\frac{1}{2}\}}$ in $\hat{u}$ and $\hat{F}$, respectively, is as follows. The random variable $\frac{2\hat{Y}_s^2}{\sigma(s,\hat{X}_s)\hat{X}_s}$ is not integrable in general, as we merely assumed that the denominator $|\sigma(s,\hat{X}_s)\hat{X}_s|$ is bounded above by $\overline{\sigma}|\hat{X}_s|$. This problem can be overcome by simply multiplying the indicator function $\mathbb{1}_{\{\hat{X}_s\ge\frac{S_0}{2}\}}$ by $\frac{2\hat{Y}_s^2}{\sigma(s,\hat{X}_s)\hat{X}_s}$. It is clear that $\mathbb{E}^{\mathbb{Q}}|\hat{u}_s|<\infty$ since*

$$|\hat{u}_s| = \frac{2\hat{Y}_s^2}{\sigma(s,\hat{X}_s)\hat{X}_s}\mathbb{1}_{\{\hat{X}_s\ge\frac{S_0}{2}\}} \le \frac{4\hat{Y}_s^2}{\underline{\sigma}S_0}$$

and $\hat{Y}_s$ is normal. The function $\hat{F}$ is also defined to satisfy this integrability condition.

Lemma 4.1.1. *Under Assumption 1, for any $p > 0$, there exists a positive constant D_p depending only on p such that the following inequalities hold.*

(i) For $0 \le t \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|u_t - \tilde{u}_t|^p] \le D_p t^p, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{u}_t - \hat{u}_t|^p] \le D_p t^p. \tag{4.1.4}$$

(ii) For $0 \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|TF - T\tilde{F}|^p] \le D_p T^p, \quad \mathbb{E}^{\mathbb{Q}}[|T\tilde{F} - T\hat{F}|^p] \le D_p T^p. \tag{4.1.5}$$

Proposition 4.1.2. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\delta(\tilde{u})\tilde{F}\right] + \mathcal{O}(\sqrt{T}) \end{aligned} \tag{4.1.6}$$

and

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\delta(\tilde{u})\tilde{F}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right] + \mathcal{O}(\sqrt{T}). \end{aligned} \tag{4.1.7}$$

Proof. We only present the proof for Eq.(4.1.6). As Eq.(4.1.7) can be proved in exactly the same manner, its proof is omitted. We may assume that $\Phi(0) = 0$. Consider a translation $\Psi(\cdot) := \Phi(\cdot) - \Phi(0)$ otherwise. Observe that

$$\begin{aligned} &\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\delta(\tilde{u})\tilde{F}\right]\right| \\ &\le \frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right| \frac{1}{\sqrt{T}}|\delta(u)|\,|TF|\right] \\ &+ \frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right| \frac{1}{\sqrt{T}}|\delta(u-\tilde{u})|\,|TF|\right] \end{aligned}$$

$$+\frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{\sqrt{T}}|\delta(\tilde{u})|\,|TF-T\tilde{F}|\right].$$

Using Assumption 3, the Hölder inequality, the Jensen inequality, and Lemma 2.0.1, for $0\le T\le 1$, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)-\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{\sqrt{T}}|\delta(u)|\,|TF|\right]\\
&\le\frac{\beta}{\sqrt{T}}\left(\frac{1}{T}\int_0^T\mathbb{E}^{\mathbb{Q}}[|X_t-\tilde{X}_t|^4]\,dt\right)^{\frac14}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{\sqrt{T}}\delta(u)\right)^2\right]\right)^{\frac12}\left(\mathbb{E}^{\mathbb{Q}}[|TF|^4]\right)^{\frac14}\\
&\le\frac{2\beta}{\underline{\sigma}}\left(\frac{B_4}{5}\right)^{\frac14}\left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le s\le T}\left(Y_s^4X_s^{-2}\right)\right]\right)^{\frac12}\left(\mathbb{E}^{\mathbb{Q}}[|TF|^4]\right)^{\frac14}\sqrt{T}\,,
\end{aligned}$$

for a positive constant B_4 defined in Lemma 2.0.1. The last inequality is based on the fact the Skorokhod integral of u becomes the Ito integral since $(u_s)_{0\le s\le T}$ is adapted to the Brownian filtration $(\mathcal{F}_s^W)_{0\le s\le T}$. Observe from $\Phi(0)=0$ and Assumption 3 that $|\Phi(x)|\le\beta|x|$ holds for any $x\in\mathbb{R}$. Thus, by similar arguments, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{\sqrt{T}}|\delta(u-\tilde{u})|\,|TF|\right]\\
&\le\frac{\beta}{\sqrt{T}}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T\tilde{X}_t\,dt\right)^4\right]\right)^{\frac14}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{\sqrt{T}}\delta(u-\tilde{u})\right)^2\right]\right)^{\frac12}\left(\mathbb{E}^{\mathbb{Q}}[|TF|^4]\right)^{\frac14}\\
&\le\beta\left(\frac{D_2}{3}\right)^{\frac12}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T\tilde{X}_t\,dt\right)^4\right]\right)^{\frac14}\left(\mathbb{E}^{\mathbb{Q}}[|TF|^4]\right)^{\frac14}\sqrt{T}\,,
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{\sqrt{T}}|\delta(\tilde{u})|\,|TF-T\tilde{F}|\right]$$

$$\le \frac{\beta}{\sqrt{T}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)^4\right]\right)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}}\Big[\left(\frac{1}{\sqrt{T}}\delta(\tilde{u})\right)^2\Big]\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}}[|TF - T\tilde{F}|^4]\right)^{\frac{1}{4}}$$

$$\le \frac{2\beta}{\underline{\sigma}}(D_4)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)^4\right]\right)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le s\le T}\left(\tilde{Y}_s^4\tilde{X}_s^{-2}\right)\Big]\right)^{\frac{1}{2}} \sqrt{T}\,,$$

for $0 \le T \le 1$, where the constants D_2, D_4 are as given in Lemma 4.1.1. Therefore, from Lemma 2.0.2, we obtain the inequality

$$\limsup_{T\to 0} \frac{1}{\sqrt{T}} \left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\delta(u)F\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\delta(\tilde{u})\tilde{F}\right]\right|$$

$$\le \frac{2\beta}{\underline{\sigma}S_0}\left(\frac{B_4}{5}\right)^{\frac{1}{4}} + S_0\beta\left(\frac{D_2}{3}\right)^{\frac{1}{2}} + \frac{2\beta}{\underline{\sigma}}(D_4)^{\frac{1}{4}} < \infty\,.$$

This gives the desired result. □

Now, we will approximate Eq.(4.1.3). To do this, we approximate D_sF by $D_s\tilde{F}$ and $D_s^*\hat{F}$, where $D_s^*\hat{F}$ is defined as

$$D_s^*\hat{F} := D_s\left(\frac{1}{\int_0^T \hat{Y}_t\,dt}\right)\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \ge \frac{1}{2}\}}\,.$$

Through some auxiliary steps, these approximations among D_sF, $D_s\tilde{F}$, and $D_s^*\hat{F}$ are presented in Lemma 4.1.3. Before investigating them, we first show in Lemma 4.1.2 that the pth moments of D_sF, $D_s\tilde{F}$, and $D_s^*\hat{F}$ are $\mathcal{O}(\frac{1}{T^p})$ in a short-maturity regime. As a necessary step, we also observe that the moments of D_sX_t, D_sY_t, $D_s\tilde{Y}_t$, and $D_s\hat{Y}_t$ are bounded. See Appendices C.3 and C.4 for the proofs.

Lemma 4.1.2. *Under Assumption 1, for any $p > 0$, there exists a positive constant E_p depending only on p such that the following inequalities hold.*

(i) For $0 \leq t \leq 1$,

$$\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|D_s X_t|^p] \leq E_p. \tag{4.1.8}$$

(ii) For $0 \leq t \leq 1$,

$$\begin{aligned} &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|D_s Y_t|^p] \leq E_p, \quad \sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|D_s \tilde{Y}_t|^p] \leq E_p, \\ &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|D_s \hat{Y}_t|^p] \leq E_p. \end{aligned} \tag{4.1.9}$$

(iii) For $0 \leq T \leq 1$,

$$\begin{aligned} &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|T D_s F|^p] \leq E_p, \quad \sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|T D_s \tilde{F}|^p] \leq E_p, \\ &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|T D_s^* \hat{F}|^p] \leq E_p. \end{aligned} \tag{4.1.10}$$

Lemma 4.1.3. *Under Assumption 1, for any* $p > 0$, *there exists a positive constant* F_p *depending only on* p *such that the following inequalities hold.*

(i) For $0 \leq t \leq 1$,

$$\begin{aligned} &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|D_s Y_t - D_s \tilde{Y}_t|^p] \leq F_p t^{\frac{p}{2}}, \\ &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|D_s \tilde{Y}_t - D_s \hat{Y}_t|^p] \leq F_p t^{\frac{p}{2}}. \end{aligned} \tag{4.1.11}$$

(ii) For $0 \leq T \leq 1$,

$$\begin{aligned} &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|T D_s F - T D_s \tilde{F}|^p] \leq F_p T^{\frac{p}{2}}, \\ &\sup_{s\geq 0} \mathbb{E}^{\mathbb{Q}}[|T D_s \tilde{F} - T D_s^* \hat{F}|^p] \leq F_p T^{\frac{p}{2}}. \end{aligned} \tag{4.1.12}$$

Proposition 4.1.3. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_sF)\,ds\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\int_0^T \tilde{u}_s(D_s\tilde{F})\,ds\right] + \mathcal{O}(\sqrt{T})\end{aligned} \tag{4.1.13}$$

and

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\int_0^T \tilde{u}_s(D_s\tilde{F})\,ds\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] + \mathcal{O}(\sqrt{T}).\end{aligned} \tag{4.1.14}$$

Proof. We only present the proof for Eq.(4.1.13). Similarly, Eq.(4.1.14) follows. We may assume that $\Phi(0) = 0$. Observe that

$$\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\int_0^T u_s(D_sF)\,ds\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\int_0^T \tilde{u}_s(D_s\tilde{F})\,ds\right]\right|$$

$$\begin{aligned}&\le \mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{T}\int_0^T |u_s||TD_sF|\,ds\right]\\ &+ \mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{T}\int_0^T |u_s - \tilde{u}_s||TD_sF|\,ds\right]\\ &+ \mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{T}\int_0^T |\tilde{u}_s||TD_sF - TD_s\tilde{F}|\,ds\right].\end{aligned}$$

Note from Assumption 3, Lemmas 2.0.1 and 4.1.2, the Hölder inequality, and the Jensen inequality that for $0 \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T \tilde{X}_t\,dt\right)\right|\frac{1}{T}\int_0^T |u_s||TD_sF|\,ds\right]$$

$$
\begin{aligned}
&\leq \beta \left(\frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^2]\, dt \right)^{\frac{1}{2}} \left(\frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}}[|u_s|^2 |TD_sF|^2]\, ds \right)^{\frac{1}{2}} \\
&\leq \beta \left(\frac{B_2}{3} \right)^{\frac{1}{2}} T \left(\frac{1}{T} \int_0^T (\mathbb{E}^{\mathbb{Q}}[|u_s|^4])^{\frac{1}{2}} (E_4)^{\frac{1}{2}}\, ds \right)^{\frac{1}{2}} \\
&\leq \frac{2\beta}{\underline{\sigma}} \left(\frac{B_2}{3} \right)^{\frac{1}{2}} (E_4)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0 \leq s \leq T} Y_s^8 X_s^{-4} \Big] \right)^{\frac{1}{4}} T
\end{aligned}
$$

holds with some positive constants B_2 and E_4. By assuming that $\Phi(0) = 0$, we obtain $|\Phi(x)| \leq \beta|x|$. From this inequality with Lemmas 4.1.1 and 4.1.2, the Hölder inequality, and the Jensen inequality, we observe that for $0 \leq T \leq 1$,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}} \left[\left| \Phi \left(\frac{1}{T} \int_0^T \tilde{X}_t\, dt \right) \right| \frac{1}{T} \int_0^T |u_s - \tilde{u}_s| |TD_sF|\, ds \right] \\
&\leq \beta \left(\mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{T} \int_0^T \tilde{X}_t\, dt \right)^2 \right] \right)^{\frac{1}{2}} \left(\frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}}[|u_s - \tilde{u}_s|^2 |TD_sF|^2]\, ds \right)^{\frac{1}{2}} \\
&\leq \beta \left(\mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{T} \int_0^T \tilde{X}_t\, dt \right)^2 \right] \right)^{\frac{1}{2}} \left(\frac{1}{T} \int_0^T (D_4)^{\frac{1}{2}} s^2 (E_4)^{\frac{1}{2}}\, ds \right)^{\frac{1}{2}} \\
&\leq \beta (D_4)^{\frac{1}{4}} (E_4)^{\frac{1}{4}} \left(\frac{1}{3} \right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{T} \int_0^T \tilde{X}_t\, dt \right)^2 \right] \right)^{\frac{1}{2}} T
\end{aligned}
$$

with some positive constants D_4, E_4. Similarly,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}} \left[\left| \Phi \left(\frac{1}{T} \int_0^T \tilde{X}_t\, dt \right) \right| \frac{1}{T} \int_0^T |\tilde{u}_s| |TD_sF - TD_s\tilde{F}|\, ds \right] \\
&\leq \beta \left(\mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{T} \int_0^T \tilde{X}_t\, dt \right)^2 \right] \right)^{\frac{1}{2}} \left(\frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}}[|\tilde{u}_s|^2 |TD_sF - TD_s\tilde{F}|^2]\, ds \right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta \left(\mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{T} \int_0^T \tilde{X}_t \, dt \right)^2 \right] \right)^{\frac{1}{2}} \left(\frac{1}{T} \int_0^T (\mathbb{E}^{\mathbb{Q}}[|\tilde{u}_s|^4])^{\frac{1}{2}} (F_4)^{\frac{1}{2}} T \, ds \right)^{\frac{1}{2}} \\
&\leq \frac{2\beta}{\underline{\sigma}} (F_4)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{T} \int_0^T \tilde{X}_t \, dt \right)^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}} \left[\max_{0 \leq s \leq T} \tilde{Y}_s^8 \tilde{X}_s^{-4} \right] \right)^{\frac{1}{4}} \sqrt{T}
\end{aligned}
$$

for a constant F_4 from Lemma 4.1.3. Therefore, from Lemma 2.0.2, we can obtain

$$
\begin{aligned}
&\limsup_{T \to 0} \frac{1}{\sqrt{T}} \Bigg| \mathbb{E}^{\mathbb{Q}} \left[\Phi \left(\frac{1}{T} \int_0^T X_t \, dt \right) \int_0^T u_s (D_s F) \, ds \right] \\
&- \mathbb{E}^{\mathbb{Q}} \left[\Phi \left(\frac{1}{T} \int_0^T \tilde{X}_t \, dt \right) \int_0^T \tilde{u}_s (D_s \tilde{F}) \, ds \right] \Bigg| \\
&\leq \frac{2\beta}{\underline{\sigma}} (F_4)^{\frac{1}{4}} < \infty \, .
\end{aligned}
$$

This gives the desired result. □

4.2 Short-maturity asymptotic for the Asian delta value

Let us concatenate the approximations established in Propositions 4.1.2 and 4.1.3 to obtain

$$
\begin{aligned}
\Delta_A(T) = \mathbb{E}^{\mathbb{Q}} &\left[\Phi \left(\frac{1}{T} \int_0^T \hat{X}_t \, dt \right) \delta(\hat{u}) \hat{F} \right] \\
&- \mathbb{E}^{\mathbb{Q}} \left[\Phi \left(\frac{1}{T} \int_0^T \hat{X}_t \, dt \right) \int_0^T \hat{u}_s (D_s^* \hat{F}) \, ds \right] + \mathcal{O}(\sqrt{T}) \, .
\end{aligned}
$$

To deduce the short-maturity asymptotic formula presented in Theorem 4.2.1, we now estimate the following two expectations:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right],\\
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right].
\end{aligned} \tag{4.2.1}$$

In the first step, $\delta(\hat{u})$ is approximated to a normal variable $\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)$ in Lemma 4.2.1. See Appendix C.5 for the proof.

Lemma 4.2.1. *Under Assumption 1, for any $p>0$, there exists a positive constant G_p depending only on p such that the following inequalities hold for $0\le T\le 1$:*

$$\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{u})|^p]\le G_pT^{\frac{p}{2}},\quad \mathbb{E}^{\mathbb{Q}}\left[\left|\delta(\hat{u})-\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right|^p\right]\le G_pT^p. \tag{4.2.2}$$

Then, using this lemma, we can directly estimate the expectations in Eq.(4.2.1) only in terms of multivariate normal random variables. Consequently, we propose the following asymptotic relations in Proposition 4.2.1. Further details are provided in Appendix C.6.

Proposition 4.2.1. *Under Assumptions 1 and 3, as $T\to 0$, we have*

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right]\\
&=\mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0+S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right]\\
&-2\frac{\Phi(S_0)}{S_0}\frac{1}{T^2}\int_0^T\frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds+\mathcal{O}(\sqrt{T})
\end{aligned} \tag{4.2.3}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] \\
&= -2\frac{\Phi(S_0)}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T})\,,
\end{aligned} \tag{4.2.4}$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

We now reach one of the two main results in this chapter. The two estimates in Proposition 4.2.1 directly give the short-maturity asymptotic for the Asian delta value $\Delta_A(T)$. Recall the definition of the Asian volatility $\sigma_A(T)$ in Eq.(3.0.1).

Theorem 4.2.1. *Under Assumptions 1 and 3, as $T \to 0$, we have*

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})$$

or, equivalently,

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z) - \Phi(S_0)}{S_0\sigma_A(T)\sqrt{T}Z}Z^2\right] + \mathcal{O}(\sqrt{T})\,,$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

Corollary 4.2.2. *Under Assumptions 1 and 3,*

(i) if $\Delta_A(T)$ converges as $T \to 0$, then

$$\lim_{T\to 0}\Delta_A(T) = \lim_{\epsilon\searrow 0}\mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0+\epsilon Z)-\Phi(S_0)}{\epsilon Z}Z^2\right]\,,$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$;

(ii) if both the right derivative $D\Phi(S_0+)$ and the left derivative $D\Phi(S_0-)$ exist, then $\Delta_A(T)$ converges and

$$\lim_{T\to 0}\Delta_A(T) = \frac{D\Phi(S_0+) + D\Phi(S_0-)}{2}\,.$$

We present some examples of Theorem 4.2.1.

Example 4.2.1. *Let* Δ_A^{call} *and* Δ_A^{put} *be the Asian call and put delta value with the strike* K*, i.e., the payoff functions are* $\Phi(x) = (x-K)_+$ *and* $\Phi(x) = (K-x)_+$*, respectively. Then,*

$$\Delta_A^{\mathrm{call}}(T) = \begin{cases} 0 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 < K\,, \\ \frac{1}{2} + \mathcal{O}(\sqrt{T}), & \text{if } S_0 = K\,, \\ 1 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 > K\,, \end{cases}$$

$$\Delta_A^{\mathrm{put}}(T) = \begin{cases} -1 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 < K\,, \\ \frac{1}{2} + \mathcal{O}(\sqrt{T}), & \text{if } S_0 = K\,, \\ 0 + \mathcal{O}(\sqrt{T}), & \text{if } S_0 > K\,. \end{cases}$$

Example 4.2.2. *Given any* $K\,, \delta > 0$*, and* $0 \le \epsilon < 1\,,$ *define the payoff function* Φ *by*

$$\Phi(x) = (x-K)^{1+\epsilon}\mathbb{1}_{\{K \le x < K+\delta\}} + \delta^{1+\epsilon}\mathbb{1}_{\{K+\delta \le x\}}\,.$$

Suppose that $S_0 = K$*. Then, we get the following asymptotic equation.*

$$\Delta_A(T) = \frac{1}{2}(S_0\sigma_A(T))^{\epsilon}\, M(2+\epsilon)T^{\frac{\epsilon}{2}} + \mathcal{O}(\sqrt{T})\,,$$

where $M(2+\epsilon) := \mathbb{E}^{\mathbb{Q}}[|Z|^{2+\epsilon}]$ *with a standard normal variable* Z*. In this example, the leading order of* $\Delta_A(T)$ *is* $T^{\frac{\epsilon}{2}}$ *as* $T \to 0$*.*

4.3 Short-maturity asymptotic for the European delta value

In the remainder of this chapter, we will investigate the short-maturity behavior of the European delta value $\Delta_E(T)$. The desired asymptotic formula is presented in Theorem 4.3.1. To prove this, we follow the same approximation steps used to derive the asymptotic formula for the Asian delta value $\Delta_A(T)$ with a slight modification. First, we use Malliavin calculus to rewrite the European delta value as the weighted sum of the payoffs.

Proposition 4.3.1. *For the process X stated in Eq.*(2.0.2) *under Assumption 1, we have*

$$\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] = \frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(X_T\right)\delta\left(\frac{Y_{\cdot}}{\sigma(\cdot,X_{\cdot})X_{\cdot}}\right)\frac{X_T}{Y_T}\right] - \frac{1}{S_0}\,\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)]$$

$$+\frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\int_0^T \frac{Y_s}{\sigma(s,X_s)X_s}\frac{X_T(D_sY_T)}{{Y_T}^2}ds\right],$$

where $\delta(\cdot)$ denotes the Skorokhod integral in $[0,T]$ and $D_s(\cdot)$ is the Malliavin derivative.

Proof. See Nualart (1995); Benhamou (2000). □

Thus, we can observe from Corollary 3.0.2 and Lemma 4.0.1 that for small $T>0$,

$$\begin{aligned}\Delta_E(T) &= \frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\Big[\Phi(X_T)\delta(h)G\Big] - \frac{\Phi(S_0)}{S_0}\\ &\quad+\frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\int_0^T h_sH_s\,ds\right] + \mathcal{O}(\sqrt{T}),\end{aligned}\tag{4.3.1}$$

where the processes $(h_s)_{0\le s\le T}$, $(H_s)_{0\le s\le T}$ and a random variable G are defined by

$$h_s := \frac{Y_s}{\sigma(s,X_s)X_s}\,,\quad H_s := \frac{X_T(D_sY_T)}{{Y_T}^2}\,,\quad G := \frac{X_T}{Y_T}\,.$$

In the second step, we approximate the two expectations on the right-hand side of Eq.(4.3.1), i.e.,

$$\frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\Big[\Phi(X_T)\delta(h)G\Big],\quad \frac{1}{S_0T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\int_0^T h_sH_s\,ds\right].\tag{4.3.2}$$

For the approximation, we define four auxiliary processes $(\tilde{h}_s)_{0\le s\le T}$, $(\hat{h}_s)_{0\le s\le T}$, $(\tilde{H}_s)_{0\le s\le T}$, $(\hat{H}_s)_{0\le s\le T}$ by

$$\tilde{h}_s := \frac{\tilde{Y}_s}{\sigma(s,\tilde{X}_s)\tilde{X}_s}\,, \quad \hat{h}_s := \frac{\hat{Y}_s}{\sigma(s,\hat{X}_s)\hat{X}_s}\mathbb{1}_{\{\hat{X}_s\ge\frac{S_0}{2}\}}$$

and

$$\tilde{H}_s := \frac{\tilde{X}_T(D_s\tilde{Y}_T)}{\tilde{Y}_T^2}\,, \quad \hat{H}_s := \frac{\hat{X}_T(D_s\hat{Y}_T)}{\hat{Y}_T^2}\mathbb{1}_{\{\hat{Y}_T\ge\frac{1}{2}\}}\,.$$

In Lemma 4.3.1, $(\tilde{h}_s)_{0\le s\le T}$, $(\hat{h}_s)_{0\le s\le T}$ are used to approximate $(h_s)_{0\le s\le T}$. Similarly, $(\tilde{H}_s)_{0\le s\le T}$, $(\hat{H}_s)_{0\le s\le T}$ are used to approximate $(H_s)_{0\le s\le T}$. We also define two new random variables $\tilde{G}$, $\hat{G}$ by

$$\tilde{G} := \frac{\tilde{X}_T}{\tilde{Y}_T}\,, \quad \hat{G} := \frac{\hat{X}_T}{\hat{Y}_T}\mathbb{1}_{\{\hat{Y}_T\ge\frac{1}{2}\}}$$

to approximate G.

Lemma 4.3.1. *Under Assumption 1, for any $p > 0$, there exists a positive constant I_p depending only on p such that the following inequalities hold.*

(i) For $0 \le t \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|h_t - \tilde{h}_t|^p] \le I_p t^p\,, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{h}_t - \hat{h}_t|^p] \le I_p t^p\,.$$

(ii) For $0 \le T \le 1$,

$$\sup_{s\ge 0}\mathbb{E}^{\mathbb{Q}}[|H_s - \tilde{H}_s|^p] \le I_p T^{\frac{p}{2}}\,, \quad \sup_{s\ge 0}\mathbb{E}^{\mathbb{Q}}[|\tilde{H}_s - \hat{H}_s|^p] \le I_p T^{\frac{p}{2}}\,.$$

(iii) For $0 \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|G - \tilde{G}|^p] \le I_p T^p\,, \quad \mathbb{E}^{\mathbb{Q}}[|\tilde{G} - \hat{G}|^p] \le I_p T^p\,.$$

Using this lemma, we can approximate the two expectations in Eq.(4.3.2). The approximation results are given in the following proposition.

Proposition 4.3.2. *Under Assumptions 1 and 3, as $T \to 0$, we have*

(i) $\displaystyle \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\Big[\Phi(X_T)\delta(h)G\Big] = \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\Big[\Phi(\tilde{X}_T)\delta(\tilde{h})\tilde{G}\Big] + \mathcal{O}(\sqrt{T}).$

(ii) $\displaystyle \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\Big[\Phi(\tilde{X}_T)\delta(\tilde{h})\tilde{G}\Big] = \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\Big[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\Big] + \mathcal{O}(\sqrt{T}).$

(iii) $\displaystyle \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T)\int_0^T h_s H_s\, ds\right] = \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\tilde{X}_T)\int_0^T \tilde{h}_s \tilde{H}_s\, ds\right] + \mathcal{O}(\sqrt{T}).$

(iv) $\displaystyle \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\tilde{X}_T)\int_0^T \tilde{h}_s \tilde{H}_s\, ds\right] = \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s \hat{H}_s\, ds\right] + \mathcal{O}(\sqrt{T}).$

Since the proofs for Lemma 4.3.1 and Proposition 4.3.2 are obtained by merely duplicating those for Lemma 4.1.1 and Propositions 4.1.2, 4.1.3, we omit them.

Then, it is easy to observe from Proposition 4.3.2 that for small $T > 0$,

$$\begin{aligned}\Delta_E(T) &= \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\Big[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\Big] - \frac{\Phi(S_0)}{S_0} \\ &\quad + \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s \hat{H}_s\, ds\right] + \mathcal{O}(\sqrt{T})\,.\end{aligned} \tag{4.3.3}$$

To obtain the asymptotic formula for $\Delta_E(T)$, we need to estimate the following two expectations for the last step.

$$\frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\Big[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\Big]\,, \quad \frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s \hat{H}_s\, ds\right]. \tag{4.3.4}$$

As a necessary lemma, we approximate $\delta(\hat{h})$ to a normal random variable $\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)$ in the following. The proof is similar to the proof for Lemma 4.2.1; hence, it is omitted.

Lemma 4.3.2. *Under Assumption 1, for any* $p > 0$, *there exists a positive constant* J_p *depending only on* p *such that the following inequalities hold for* $0 \leq T \leq 1$*:*

$$\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{h})|^p] \leq J_p T^{\frac{p}{2}}, \quad \mathbb{E}^{\mathbb{Q}}\left[\left|\delta(\hat{h}) - \delta\left(\frac{1}{\sigma(\cdot, S_0)S_0}\right)\right|^p\right] \leq J_p T^p .$$

As Lemma 4.2.1 helps estimate the expectations in Eq.(4.2.1), this lemma enables us to estimate the expectations in Eq.(4.3.4) in relation to multivariate normal random variables. Direct calculations with regard to normal random variables yield the following two estimates. See Appendix C.7 for details.

Proposition 4.3.3. *Under Assumptions 1 and 3, as* $T \to 0$, *we have*

$$\begin{aligned}&\frac{1}{S_0 T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] \\ &+ \frac{\Phi(S_0)}{S_0 T}\int_0^T \frac{\sigma(s, S_0) - \nu(s, S_0)}{\sigma(s, S_0)}\,ds + \mathcal{O}(\sqrt{T})\end{aligned} \tag{4.3.5}$$

and

$$\frac{1}{S_0 T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s \hat{H}_s\,ds\right] = \frac{\Phi(S_0)}{S_0 T}\int_0^T \frac{\nu(s, S_0)}{\sigma(s, S_0)}\,ds + \mathcal{O}(\sqrt{T})\,, \tag{4.3.6}$$

where Z *is a standard normal random variable, i.e.,* $Z \sim N(0, 1)$.

Therefore, we obtain the desired asymptotic formula for $\Delta_E(T)$ by straightforward use of Proposition 4.3.3 in Eq.(4.3.3).

Theorem 4.3.1. *Under Assumptions 1 and 3, as* $T \to 0$, *we have*

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})$$

or, equivalently,

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z) - \Phi(S_0)}{S_0\sigma_E(T)\sqrt{T}Z}Z^2\right] + \mathcal{O}(\sqrt{T}),$$

where Z is a standard normal random variable, i.e., $Z \sim N(0,1)$.

By comparing Theorem 4.2.1 with Theorem 4.3.1, we can observe that the limits of $\Delta_A(T)$ and $\Delta_E(T)$ are the same.

Corollary 4.3.2. *Under Assumptions 1 and 3, if $\Delta_E(T)$ converges as $T \to 0$, then $\Delta_A(T)$ also converges and vice versa. Moreover,*

$$\lim_{T\to 0} \Delta_A(T) = \lim_{T\to 0} \Delta_E(T).$$

Chapter 5

Short-maturity options with Hölder continuous payoffs

In this chapter, we generalize Theorems 3.0.1, 4.2.1, and 4.3.1 to the Hölder continuous function Φ . Under Assumption 1, the first variation process of S is the unique solution of Eq.(C.1.1):

$$dZ_t = (r-q)Z_t\,dt + \nu(t,S_t)Z_t\,dW_t\,, \quad Z_0 = 1\,.$$

Analogous to Lemma 2.0.1, the lemma below is crucial in the following.

Lemma 5.0.1. *Under Assumption 1, for any* $p > 0$, *as* $t \to 0$, *we have*

$$\mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^p] = \mathcal{O}(t^p)\,, \quad \mathbb{E}^{\mathbb{Q}}[|Z_t - Y_t|^p] = \mathcal{O}(t^p)\,.$$

Proof. The first equality with $p=2$ has already been proved in the proof of Lemma 3.0.1 in Appendix B.1. The remainder of the proof is the same as that of Lemma 2.0.1 in Appendix A.1. □

This chapter considers Hölder continuous payoffs in the following order. In Section 5.1, the asymptotic for option prices in Theorem 3.0.1 will be generalized. Estimates for the option delta value in Theorems 4.2.1 and 4.3.1 are generalized to Hölder continuous payoffs in Section 5.2.

5.1 Estimates for option prices

The following lemma is a generalization of Lemma 3.0.1.

Lemma 5.1.1. *Under Assumptions 1 and 2, as $T \to 0$, we have*

(i) $P_A(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] + \mathcal{O}(T^{\gamma})\,,$

(ii) $P_E(T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] + \mathcal{O}(T^{\gamma})\,.$

Here, γ is the Hölder exponent in Assumption 2.

Proof. Choose $q > 1$ such that $\gamma q > 1$. Then, by the Jensen inequality and Assumption 2,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right) - \Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] \\
&\le \beta\,\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |S_t - X_t|\,dt\right)^{\gamma}\right] \le \beta\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^{\gamma q}]\,dt\right)^{\frac{1}{q}}\,.
\end{aligned}$$

Then, the remainder of the proof comes from Lemma 5.0.1. □

Now, we get the asymptotic estimates for the prices of options having Hölder continuous payoffs.

Theorem 5.1.1. *Under Assumptions 1 and 2, as $T \to 0$, we have*

$$\begin{aligned}
P_A(T) &= \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T^{\gamma})\,, \\
P_E(T) &= \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)] + \mathcal{O}(T^{\gamma})\,.
\end{aligned}$$

Here, γ is the Hölder exponent in Assumption 2.

Proof. The proof is straightforward from the proofs of Theorems 3.0.1 and 5.1.1. □

Compared to the Lipschitz continuous case ($\gamma = 1$), the convergence order is degraded for $\gamma < 1$. However, the following example shows that this asymptotic relation is optimal in general.

Example 5.1.1. *Given any K and $0 < \gamma \leq 1$, define the payoff function Φ by*

$$\Phi(x) = (x - K)_+^\gamma .$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$P_A(T) = \frac{1}{2}(S_0 \sigma_A(T))^\gamma M(\gamma) T^{\frac{\gamma}{2}} + \mathcal{O}(T^\gamma),$$

where $M(\gamma) := \mathbb{E}^{\mathbb{Q}}[|Z|^\gamma]$ with a standard normal variable Z. If we replace $\sigma_A(T)$ with $\sigma_E(T)$, we get the asymptotic result for the European option price $P_E(T)$.

5.2 Estimates for option delta values

In this section, we investigate the short-maturity option delta values when Φ is any Hölder continuous function. The main results are as follows:

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}} Z\right] + \mathcal{O}(T^{\gamma - \frac{1}{2}})$$

and

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}} Z\right] + \mathcal{O}(T^{\gamma - \frac{1}{2}}),$$

where Z denotes a standard normal variable and γ is the Hölder exponent in Assumption 2.

The proof begins by recognizing the changes in Lemma 4.0.1 from Chapter 4. In the proof of Lemma 4.0.1, we make use of the fact that any Lipschitz continuous function is almost everywhere differentiable with respect to the Lebesgue measure. However, this condition fails for arbitrary γ-Hölder continuous functions in general. Therefore, we should rely only on the Malliavin representation of the option delta (see Proposition 4.1.1) for the approximation. First, we examine the following Malliavin representation for $\Delta_A(T)$.

Proposition 5.2.1. *For the process S stated in Eq.*(2.0.1)*, under Assumption 1, we have*

$$\Delta_A(T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\delta\left(\frac{2Z_\cdot^{\,2}}{\sigma(\cdot,S_\cdot)S_\cdot}\right)\frac{1}{\int_0^T Z_t\,dt}\right] - e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\int_0^T \frac{2{Z_u}^2}{\sigma(u,S_u)S_u}D_u\left(\frac{1}{\int_0^T Z_t\,dt}\right)du\right],$$

where $\delta(\cdot)$ is the Skorokhod integral in $[0,T]$ and $D_s(\cdot)$ is the Malliavin derivative.

Proof. See Nualart (1995); Benhamou (2000). □

As we approximated u by $\tilde{u}$ and $\hat{u}$ in Lemma 4.1.1, we would approximate a process $\frac{2Z_\cdot^2}{\sigma(\cdot,S_\cdot)S_\cdot}$ by u . Likewise, a random variable $\frac{1}{\int_0^T Z_t\,dt}$ is approximated to F . Examine following Lemmas 5.2.1 and 5.2.2 in comparison to Lemmas 4.1.1, 4.1.2, and 4.1.3.

Lemma 5.2.1. *Under Assumption 1, for any $p > 0$, as $t \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2Z_t^2}{\sigma(t,S_t)S_t} - u_t\right|^p\right] = \mathcal{O}(t^p).$$

Proof. Same as the proof of Lemma 4.1.1. Use Lemma 5.0.1 instead of Lemma 2.0.1. □

Lemma 5.2.2. *Under Assumption 1, for any $p > 0$, as $T \to 0$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{1}{\frac{1}{T}\int_0^T Z_t\,dt} - TF\right|^p\right] = \mathcal{O}(T^p)\,,$$
$$\sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}\left[\left|D_s\left(\frac{1}{\frac{1}{T}\int_0^T Z_t\,dt}\right) - TD_sF\right|^p\right] = \mathcal{O}(T^p)\,.$$

Proof. Examine the following Malliavin derivatives:

$$D_u S_l = \frac{Z_l}{Z_u}\sigma(u, S_u)S_u\mathbb{1}_{\{u\leq l\}}\,,$$

$$D_u Z_t = Z_t\left[\nu(u, S_u) - \int_0^t \nu(l, S_l)\rho(l, S_l)D_u S_l\,dl + \int_0^t \rho(l, S_l)D_u S_l\,dW_l\right]\mathbb{1}_{\{u\leq t\}}\,.$$

The remainder of the proof is similar to the proofs of Lemmas 4.1.1–4.1.3. Use Lemma 5.0.1 instead of Lemma 2.0.1. □

Through minor changes in Propositions 4.1.2 and 4.1.3, we obtain the generalized version of Lemma 4.0.1.

Lemma 5.2.3. *Under Assumptions 1 and 2, as $T \to 0$, we have*

$$\Delta_A(T) = e^{-rT}\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T X_t\,dt\right)\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,.$$

Proof. Apply Lemmas 5.2.1 and 5.2.2 to the proofs of Propositions 4.1.2 and 4.1.3 instead of Lemmas 4.1.1–4.1.3. □

This section is devoted to the following result, which is the generalization of Theorems 4.2.1 and 4.3.1.

Theorem 5.2.1. *Under Assumptions 1 and 2, as $T \to 0$, we have*

$$\Delta_A(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})$$

and

$$\Delta_E(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,.$$

Proof. Minor changes to the convergence rates in Propositions 4.1.2–4.2.1 give the first asymptotic. For the European delta, we duplicate the arguments in this section. □

Remark 5.2.1. *The borderline is $\gamma = \frac{1}{2}$ in these formulas. If $\gamma < \frac{1}{2}$, the estimates in Theorem 5.2.1 are meaningless; however, for $\frac{1}{2} < \gamma \leq 1$, they provide us with the short-maturity estimate with the convergence rate $\gamma - \frac{1}{2} \leq \frac{1}{2}$.*

Example 5.2.1. *Given any K and $\frac{1}{2} < \gamma < 1$, define the payoff function Φ by*

$$\Phi(x) = (x-K)_+^{\gamma}\,.$$

Suppose that $S_0 = K$. Then, we get the following asymptotic equation.

$$\Delta_A(T) = \frac{M(1+\gamma)}{2(S_0\sigma_A(T))^{1-\gamma}} \times \frac{1}{T^{\frac{1-\gamma}{2}}} + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,,$$

where $M(1+\gamma) := \mathbb{E}^{\mathbb{Q}}[|Z|^{1+\gamma}]$ with a standard normal variable Z.

Chapter 6

Comparison between volatilities at short maturity

To emphasize the dependence on the payoff function Φ and the volatility function $\sigma(t,x)$, we denote the option price $P_A(T)$, $P_E(T)$ and the option delta value $\Delta_A(T)$, $\Delta_E(T)$ by $P_A(T;\Phi\,,\sigma)$, $P_E(T;\Phi\,,\sigma)$ and $\Delta_A(T;\Phi\,,\sigma)$, $\Delta_E(T;\Phi\,,\sigma)$, respectively. Since the Asian and European volatilities $\sigma_A(T)$, $\sigma_E(T)$ defined in Eq.(3.0.1) also depend on the volatility function $\sigma(t,x)$, we denote them by $\sigma_A(T;\sigma)$, $\sigma_E(T;\sigma)$.

6.1 Comparison under the general local volatility model

In practice, Asian-style options are mainly quoted by their prices, not by their implied volatilities. This is due to the lack of a simple closed-form formula for the density of $\frac{1}{T}\int_0^T S_t\,dt$. Instead, many practitioners estimate the implied volatilities of European-style options for pricing and hedging purposes.

Let us focus on the situation in which the volatility function $\sigma(t,x)$ of the underlying process Eq.(2.0.1) is calibrated to match market data on

European-style options. Denote this calibrated volatility by $\sigma_{\text{Implied}}(t,x)$. In this section, we aim to price and hedge the Asian option under the volatility function $\sigma_{\text{Implied}}(t,x)$. Note from the asymptotic formulas established in Theorems 3.0.1, 4.2.1, and 4.3.1 that if a function $\tau(t,x)$ satisfies Assumption 1 and the equality

$$\sigma_A(T;\sigma_{\text{Implied}}) = \sigma_E(T;\tau) \tag{6.1.1}$$

for sufficiently small $T > 0$, then for any Lipschitz continuous function $\Phi(\cdot)$,

$$\begin{aligned} P_A(T;\Phi,\sigma_{\text{Implied}}) &= P_E(T;\Phi,\tau) + \mathcal{O}(T)\,, \\ \Delta_A(T;\Phi,\sigma_{\text{Implied}}) &= \Delta_E(T;\Phi,\tau) + \mathcal{O}(\sqrt{T})\,. \end{aligned} \tag{6.1.2}$$

Meanwhile, if the equality Eq.(6.1.1) fails in any neighborhood of $T = 0$ and, heuristically speaking, τ deviates excessively from σ_{Implied}, we will see in Propositions 6.1.1 and 6.1.2 that the convergence rates in Eq.(6.1.2) are degraded in general. In this sense, we may regard the European option under the volatility $\tau(t,x)$ satisfying Eq.(6.1.1) as a short-maturity proxy for the Asian option under the volatility $\sigma_{\text{Implied}}(t,x)$.

Proposition 6.1.1. *Consider any volatility function τ that satisfies Assumption 1.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$P_A(T;\Phi,\sigma_{\text{Implied}}) = P_E(T;\Phi,\tau) + \mathcal{O}(\sqrt{T})\,.$$

(ii) Suppose that $\limsup_{T\to 0} |\sigma_A(T;\sigma_{\text{Implied}}) - \sigma_E(T;\tau)| \neq 0$. Then, for any $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|P_A(T;\Phi_\epsilon,\sigma_{\text{Implied}}) - P_E(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^{\frac{1}{2}+\epsilon})\,. \tag{6.1.3}$$

(iii) By contrast, suppose that $\sigma_A(T;\sigma_{\text{Implied}}) = \sigma_E(T;\tau)$ for small $T > 0$.

Then, for any Φ that satisfies Assumption 3,

$$P_A(T;\Phi,\sigma_{\text{Implied}}) = P_E(T;\Phi,\tau) + \mathcal{O}(T)\,.$$

Proof. Except for Eq.(6.1.3), the remainder of the proof is obvious from Theorem 3.0.1. Take Φ_ϵ as

$$\Phi_\epsilon(x) := (x-K)^{1+\epsilon}\mathbb{1}_{\{K\le x<2K\}} + (2K)^{1+\epsilon}\mathbb{1}_{\{2k\le x\}}\,,$$

with $K = S_0$. Then, Eq.(6.1.3) easily follows. □

Proposition 6.1.2. *Consider any volatility function τ that satisfies Assumption 1.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$\Delta_A(T;\Phi,\sigma_{\text{Implied}}) = \Delta_E(T;\Phi,\tau) \,+ \mathcal{O}(1)\,.$$

(ii) Suppose that $\limsup_{T\to 0}|\sigma_A(T;\sigma_{\text{Implied}}) - \sigma_E(T;\tau)| \ne 0$. Then, for any $0<\epsilon\le\frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|\Delta_A(T;\Phi_\epsilon,\sigma_{\text{Implied}}) - \Delta_E(T;\Phi_\epsilon,\tau)| \ne \mathcal{O}(T^\epsilon)\,.$$

(iii) By contrast, suppose that $\sigma_A(T;\sigma_{\text{Implied}}) = \sigma_E(T;\tau)$ for small $T>0$. Then, for any Φ that satisfies Assumption 3,

$$\Delta_A(T;\Phi,\sigma_{\text{Implied}}) = \Delta_E(T;\Phi,\tau) + \mathcal{O}(\sqrt{T})\,.$$

Proof. Same as that of Proposition 6.1.1. □

Remark 6.1.1. *Suppose that $s\mapsto\sigma_{\text{Implied}}(s,S_0)$ and $s\mapsto\tau(s,S_0)$ are both continuous at $s=0$. Then, the condition* $\limsup_{T\to 0}|\sigma_A(T;\sigma_{\text{Implied}}) -$

$\sigma_E(T;\tau)| \neq 0$ *is equivalent to*

$$\tau(0,S_0) \neq \frac{1}{\sqrt{3}}\sigma_{\text{Implied}}(0,S_0)\,,$$

because $\lim_{T\to 0}\sigma_A(T;\sigma_{\text{Implied}}) = \frac{1}{\sqrt{3}}\sigma_{\text{Implied}}(0,S_0)$ *and* $\lim_{T\to 0}\sigma_E(T;\tau) = \tau(0,S_0)$ *. These limits coincide with the well-known result that ATM Asian vol* $= \frac{1}{\sqrt{3}}\cdot$ *ATM European vol. See Pirjol and Zhu (2016).*

If some suitable technical condition is satisfied for $s \mapsto \sigma_{\text{Implied}}(s,S_0)$, then Eq.(6.1.1) forces τ to be determined uniquely. Hence, whenever σ_{Implied} is given, we can always approximate the Asian option having volatility σ_{Implied} by the European option having volatility τ .

Proposition 6.1.3. *Suppose that* τ *satisfies Assumption 1 and Eq.*(6.1.1)*. If* $s \mapsto \sigma_{\text{Implied}}(s,S_0)$ *is continuous in some neighborhood of* $s=0$ *, say* $[0,\delta]$ *, then* $s \mapsto \tau(s,S_0)$ *is uniquely determined in* $[0,\delta]$ *by*

$$\tau(s,S_0) = \left[\frac{2}{s^3}\int_0^s \sigma^2_{\text{Implied}}(u,S_0)(us-u^2)\,du\right]^{\frac{1}{2}}\,.$$

Conversely, if $s \mapsto \tau(s,S_0)$ *is of* $C^2[0,\delta]$*, then the only choice of a continuous function* $s \mapsto \sigma_{\text{Implied}}(s,S_0)$ *in* $[0,\delta]$ *is*

$$\sigma_{\text{Implied}}(s,S_0) = \tau(s,S_0)\left[3 + 6s\frac{\tau'(s,S_0)}{\tau(s,S_0)} + s^2\left(\frac{\tau'(s,S_0)}{\tau(s,S_0)}\right)^2 + s^2\frac{\tau''(s,S_0)}{\tau(s,S_0)}\right]^{\frac{1}{2}}\,.$$

Proof. Differentiate both sides of Eq.(6.1.1) by T . □

6.2 Comparison under the Black–Scholes model

In this section, we focus on the Black–Scholes model, i.e., $\sigma(t,x) \equiv \sigma$. Observe that under the Black–Scholes model, $\sigma_A(T;\sigma) = \frac{\sigma}{\sqrt{3}}$ and $\sigma_E(T;\sigma) =$

σ. Now, consider a new option, i.e., the so-called *geometric average Asian option*, whose price is given by

$$P_G^{\mathrm{BS}}(T;\Phi,\sigma) := e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(e^{\frac{1}{T}\int_0^T \log S_t\,dt}\right)\right].$$

Here, the superscript BS is used to emphasize the Black–Scholes model. We denote its delta value by $\Delta_G^{\mathrm{BS}}(T;\Phi,\sigma)$.

Let us confine ourselves to the situation in which a constant $\sigma_{\mathrm{Implied}}$ is obtained from the European option. We want to approximate the Asian option price $P_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}})$ and its delta value $\Delta_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}})$ by their European and geometric average Asian counterparts. In Propositions 6.2.1 and 6.2.2, we observe that the European option having volatility $\frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}$ and the geometric average Asian option having volatility $\sigma_{\mathrm{Implied}}$ are optimal choices for the asymptotic approximation.

Proposition 6.2.1. *Consider any constant volatility $\tau > 0$.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$P_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}}) = P_E^{\mathrm{BS}}(T;\Phi,\tau) + \mathcal{O}(\sqrt{T}),$$
$$P_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}}) = P_G^{\mathrm{BS}}(T;\Phi,\tau) + \mathcal{O}(\sqrt{T}).$$

(ii) Suppose that $\tau \neq \frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}$. Then, for any $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) - P_E^{\mathrm{BS}}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^{\frac{1}{2}+\epsilon}).$$

Likewise, if $\tau \neq \sigma_{\mathrm{Implied}}$ and $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) - P_G^{\mathrm{BS}}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^{\frac{1}{2}+\epsilon}).$$

(iii) By contrast, for any Φ that satisfies Assumption 3,

$$P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) = P_E^{\mathrm{BS}}\left(T;\Phi_\epsilon,\frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}\right) + \mathcal{O}(T)\,,$$
$$P_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) = P_G^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) + \mathcal{O}(T)\,.$$

Proof. Under the Black–Scholes model, $e^{\frac{1}{T}\int_0^T \log S_t\,dt}$ is a log-normal random variable. Hence, it is easy to check from Lemma 3.0.1 that

$$P_G^{\mathrm{BS}}(T;\Phi,\tau) = P_E^{\mathrm{BS}}\left(T;\Phi,\frac{1}{\sqrt{3}}\tau\right) + \mathcal{O}(T)$$

for any positive constant τ and Φ satisfying Assumption 3. The remainder of the proof is the same as that of Proposition 6.1.1. □

Proposition 6.2.2. *Consider any constant volatility $\tau > 0$.*

(i) For any payoff function Φ that satisfies Assumption 3,

$$\Delta_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}}) = \Delta_E^{\mathrm{BS}}(T;\Phi,\tau) + \mathcal{O}(1)\,,$$
$$\Delta_A^{\mathrm{BS}}(T;\Phi,\sigma_{\mathrm{Implied}}) = \Delta_G^{\mathrm{BS}}(T;\Phi,\tau) + \mathcal{O}(1)\,.$$

(ii) Suppose that $\tau \neq \frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}$. Then, for any $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|\Delta_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) - \Delta_E^{\mathrm{BS}}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^\epsilon)\,.$$

Likewise, if $\tau \neq \sigma_{\mathrm{Implied}}$ and $0 < \epsilon \leq \frac{1}{2}$, there exists a payoff function Φ_ϵ that satisfies Assumption 3 such that

$$|\Delta_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) - \Delta_G^{\mathrm{BS}}(T;\Phi_\epsilon,\tau)| \neq \mathcal{O}(T^\epsilon)\,.$$

(iii) By contrast, for any Φ that satisfies Assumption 3,

$$\Delta_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) = \Delta_E^{\mathrm{BS}}\left(T;\Phi_\epsilon,\frac{1}{\sqrt{3}}\sigma_{\mathrm{Implied}}\right) + \mathcal{O}(\sqrt{T}),$$
$$\Delta_A^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) = \Delta_G^{\mathrm{BS}}(T;\Phi_\epsilon,\sigma_{\mathrm{Implied}}) + \mathcal{O}(\sqrt{T}).$$

Proof. From Lemma 4.0.1 and Lemma C.1.1 in its proof, it is easy to check that

$$\Delta_G^{\mathrm{BS}}(T;\Phi,\tau) = \Delta_E^{\mathrm{BS}}\left(T;\Phi,\frac{1}{\sqrt{3}}\tau\right) + \mathcal{O}(\sqrt{T})$$

for any positive constant τ and Φ satisfying Assumption 3. The remainder of the proof is the same as that of Proposition 6.1.2. □

Remark 6.2.1. *Consider any positive constant τ. Even though $P_A^{\mathrm{BS}}(T;\Phi,\tau)$, $P_G^{\mathrm{BS}}(T;\Phi,\tau)$, $P_E^{\mathrm{BS}}(T;\Phi,\tau)$ as well as $\Delta_A^{\mathrm{BS}}(T;\Phi,\tau)$, $\Delta_G^{\mathrm{BS}}(T;\Phi,\tau)$, $\Delta_E^{\mathrm{BS}}(T;\Phi,\tau)$ share the same limit as $T \to 0$ from Corollaries 3.0.2 and 4.2.2, Propositions 6.2.1 and 6.2.2 argue that the Asian option is more "close" to the geometric average Asian option than to the European option.*

Chapter 7

Special case: Approximation for call and put options

This chapter only considers the Asian call option, i.e., $\Phi(x) = (x - K)_+$, and the Asian put option, i.e., $\Phi(x) = (K - x)_+$. The meanings of the following notations are self-explanatory:

$$P_A^{\text{call}}(T)\,,\ P_A^{\text{put}}(T)\,,\ \Delta_A^{\text{call}}(T)\,,\ \Delta_A^{\text{put}}(T)\,.$$

The short-maturity behaviors of these four quantities have already been examined in Examples 3.0.1 and 4.2.1. However, this chapter uses the large deviation principle to provide additional information about their behaviors.

7.1 Application of the large deviation principle

Consider the model where the volatility function $\sigma(t, x)$ is independent of t. In other words, $\sigma(t, x) \equiv \sigma(x)$. Besides Assumption 1, let us impose the following assumption on the volatility function $\sigma(x)$.

Assumption 4. *There are constants $M > 0$ and $\gamma > 0$ such that for any*

$x, y \in \mathbb{R}$,

$$|\sigma(e^x) - \sigma(e^y)| \leq M|x - y|^\gamma .$$

Under Assumptions 1 and 4, the following short-maturity asymptotic results for $P_A^{\text{call}}(T)$ and $P_E^{\text{put}}(T)$ were first proved in Pirjol and Zhu (2016).

Theorem 7.1.1 (Pirjol, D. and L. Zhu Pirjol and Zhu (2016))**.** *Under Assumptions 1 and 4, the following hold.*

(i) For an OTM Asian call option, i.e., $K > S_0$,

$$\lim_{T \to 0} T \log(P_A^{\text{call}}(T)) = -\mathcal{I}(K, S_0) .$$

(ii) For an OTM Asian put option, i.e., $S_0 > K$,

$$\lim_{T \to 0} T \log(P_A^{\text{put}}(T)) = -\mathcal{I}(K, S_0) .$$

Here, for any $x\,, y > 0$, the rate function $\mathcal{I}$ is defined by

$$\mathcal{I}(x\,, y) := \inf_{\substack{\int_0^1 e^{g(t)}\,dt = x, \\ g(0) = \log y,\, g \in \mathcal{AC}[0,1]}} \frac{1}{2} \int_0^1 \left(\frac{g^{'}(t)}{\sigma(e^{g(t)})} \right)^2 dt\,, \tag{7.1.1}$$

where $\mathcal{AC}[0,1]$ is the space of absolutely continuous functions on $[0,1]$.

Remark 7.1.1. *Note that the rate function $\mathcal{I}$ in Pirjol and Zhu (2016) comes from the large deviation principle. According to the large deviation principle, for any Borel set A in $\mathbb{R}^+$,*

$$\begin{aligned} -\inf_{x \in A^\circ} \mathcal{I}(x, S_0) &\leq \liminf_{T \to 0} T \log \left(\mathbb{Q} \left\{ \frac{1}{T} \int_0^T S_t\, dt \in A \right\} \right) \\ &\leq \limsup_{T \to 0} T \log \left(\mathbb{Q} \left\{ \frac{1}{T} \int_0^T S_t\, dt \in A \right\} \right) \leq - \inf_{x \in \overline{A}} \mathcal{I}(x, S_0)\,, \end{aligned}$$

where A° is the interior of A and $\overline{A}$ is the closure of A. See Dembo (1998); Pirjol and Zhu (2016) for details.

By solving the variational problem on the right-hand side of Eq.(7.1.1), the following property of the rate function $\mathcal{I}$ was proposed in Pirjol and Zhu (2016).

Proposition 7.1.1 (Pirjol, D. and L. Zhu Pirjol and Zhu (2016))**.** *Given any $y > 0$, $x \mapsto \mathcal{I}(x, y)$ is a continuous map that is monotone decreasing in $(-\infty, y]$ and monotone increasing in $[y, \infty)$.*

7.2 Short-maturity asymptotic for the Asian call and put option delta value

Similarly to Pirjol and Zhu (2016), we use the large deviation theory to examine the short-maturity asymptotic for $\Delta_A^{\text{call}}(T)$ and $\Delta_E^{\text{put}}(T)$. See Appendix D.1 for the proof.

Theorem 7.2.1. *Under Assumptions 1 and 4, the following hold for the rate function $\mathcal{I}$ defined by Eq.*(7.1.1)*.*

(i) For an OTM Asian call option, i.e., $K > S_0$,

$$\lim_{T\to 0} T\log(\Delta_A^{\text{call}}(T)) = -\mathcal{I}(K, S_0)\,. \tag{7.2.1}$$

(ii) For an OTM Asian put option, i.e., $S_0 > K$,

$$\lim_{T\to 0} T\log(-\Delta_A^{\text{put}}(T)) = -\mathcal{I}(K, S_0)\,. \tag{7.2.2}$$

As a corollary of Theorem 7.2.1, we can approximate ITM Asian call and put option delta values.

Corollary 7.2.2. *Under Assumptions 1 and 4, the following asymptotic relations hold as $T \to 0$.*

(i) For an ITM Asian call option, i.e., $S_0 > K$,

$$\Delta_A^{\text{call}}(T) = 1 - \frac{1}{2}(r+q)T + \left(\frac{r^2+rq+q^2}{6}\right)T^2 + \mathcal{O}(T^3)\,.$$

(ii) For an ITM Asian put option, i.e., $K > S_0$,

$$\Delta_A^{\text{put}}(T) = -1 + \frac{1}{2}(r+q)T - \left(\frac{r^2+rq+q^2}{6}\right)T^2 + \mathcal{O}(T^3)\,.$$

Proof. From Lemma C.1.1 in Appendix C.1, we can get the put-call parity for the Asian option delta value:

$$\begin{aligned}\Delta_A^{\text{call}}(T) - \Delta_A^{\text{put}}(T) &= \frac{e^{-rT}}{S_0}\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[S_t]\,dt \\ &= \frac{e^{-rT}}{S_0}\frac{1}{T}\int_0^T S_0 e^{(r-q)t}\,dt = \frac{e^{-qT}-e^{-rT}}{(r-q)T}\,.\end{aligned}$$

From Theorem 7.2.1, the OTM Asian delta value vanishes at an exponential rate. Therefore, the Taylor expansion

$$\frac{e^{-qT}-e^{-rT}}{(r-q)T} = 1 - \frac{1}{2}(r+q)T + \left(\frac{r^2+rq+q^2}{6}\right)T^2 + \mathcal{O}(T^3)$$

gives Corollary 7.2.2. □

Remark 7.2.1. *Theorem 7.2.1 and its Corollary 7.2.2 are obtained from direct use of the large deviation theory. This is different from the method used in Pirjol and Zhu (2018). More precisely, Pirjol and Zhu (2018) involved a sensitivity analysis of approximated option prices, not true option prices.*

Remark 7.2.2. *Observe that Corollary 7.2.2 extends the result in Example 4.2.1. The drift term determines the order greater than $\sqrt{T}$.*

Chapter 8

Conclusion

This paper described the short-maturity asymptotic analysis of the Asian option having an arbitrary Hölder continuous payoff in the local volatility model. We were mainly interested in the Asian option price and the Asian option delta value. The short-maturity behaviors of the option price and the delta value were both expressed in terms of the *Asian volatility*, which was defined by

$$\sigma_A(T) = \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t, S_0)(T-t)^2\, dt}\,.$$

For sufficiently small $T > 0$, we proved that

$$\begin{aligned} P_A(T) &= \mathbb{E}^{\mathbb{Q}}[\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)] + \mathcal{O}(T^\gamma)\,, \\ \Delta_A(T) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0 + S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right] + \mathcal{O}(T^{\gamma-\frac{1}{2}})\,, \end{aligned}$$

for a standard normal random variable Z and the Hölder exponent γ of the payoff function Φ.

These asymptotic results were based on the idea that an underlying process $(S_t)_{t\geq 0}$ under the local volatility model can be approximated by some suitable Gaussian processes in the $L^p(\mathbb{Q})$ norm. To implement this main idea in the approximation, we used Malliavin calculus theory to represent the delta of the Asian option. In addition, we used the large deviation principle to investigate an asymptotic for the Asian call and put option.

For comparison with the Asian option, we examined the short-maturity behavior of the European option. In contrast to the Asian volatility, we proved that at short maturity T, the European option is expressed by the *European volatility*. In terms of these volatilities, we observed the resemblance between Asian and European options at short maturity.

Appendix A

Detailed proof of Chapter 2

A.1 Proof for Lemma 2.0.1

Proof. First, we prove the second inequality of Eq.(2.0.5). It suffices to show this for $p \geq 2$ since once this is proven, for $0 < p < 2$, by the Jensen inequality,

$$\left(\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p]\right)^{\frac{2}{p}} \leq \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^2] \leq B_2 t^2$$

for some constant $B_2 > 0$. Now, for $p \geq 2$, observe that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] &\leq C_p \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^t |\sigma(s, S_0)\tilde{X}_s - \sigma(s, S_0)S_0|^2 \, ds\right)^{\frac{p}{2}}\right] \\ &\leq C_p \overline{\sigma}^p \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^t |\tilde{X}_s - S_0|^2 \, ds\right)^{\frac{p}{2}}\right] \\ &\leq C_p \overline{\sigma}^p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - S_0|^p] \, ds \end{aligned} \tag{A.1.1}$$

for some constant $C_p > 0$. For these inequalities, we used the Burkholder–Davis–Gundy inequality, Assumption 1, and the Jensen inequality. Using the Jensen inequality $(\frac{x+y}{2})^p \leq \frac{x^p+y^p}{2}$ for $x, y \geq 0$, it follows that for $t \leq 1$,

$$
\begin{aligned}
& t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - S_0|^p]\, ds \\
\leq & 2^{p-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^p]\, ds + 2^{p-1} t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\hat{X}_s - S_0|^p]\, ds \\
\leq & 2^{p-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^p]\, ds + 2^{p-1} \overline{\sigma}^p {S_0}^p M(p) \frac{1}{\frac{p}{2}+1} t^p
\end{aligned}
\tag{A.1.2}
$$

where $M(p)$ is defined as $\mathbb{E}^{\mathbb{Q}}[|Z|^p]$ while Z denotes a standard normal distribution. Because $\hat{X}_t$ is a normal random variable for each $t > 0$, direct calculation gives Eq.(A.1.2). Hence, from Eqs.(A.1.1) and (A.1.2), we get

$$
\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \leq f_p(t) + A_p \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^p]\, ds\,,
$$

where $f_p(t) := C_p \overline{\sigma}^p 2^{p-1} \overline{\sigma}^p {S_0}^p M(p) \frac{1}{\frac{p}{2}+1} t^p$ and $A_p := C_p \overline{\sigma}^p 2^{p-1}$. Then, by the Gronwall inequality, for any $0 \leq t \leq 1$,

$$
\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \leq f_p(t) + A_p \int_0^t f_p(s) e^{A_p(t-s)}\, ds\,.
$$

Thus, we can find a constant B_p for $\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^p] \leq B_p t^p$.

For the first inequality of Eq.(2.0.5), we also present the proof for $p \geq 2$. Observe that $(X_t - \tilde{X}_t)_{t \geq 0}$ is a continuous martingale starting at 0. By the Burkholder–Davis–Gundy inequality and the Jensen inequality, we get

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] &\leq C_p \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^t |\sigma(s, X_s) X_s - \sigma(s, S_0) \tilde{X}_s|^2\, ds \right)^{\frac{p}{2}} \right] \\
&\leq C_p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, X_s) X_s - \sigma(s, S_0) \tilde{X}_s|^p]\, ds
\end{aligned}
$$

for some constant $C_p > 0$. Using the Jensen inequality $(\frac{x+y+z}{3})^p \leq \frac{x^p+y^p+z^p}{3}$ for $x, y, z \geq 0$,

$$
\begin{aligned}
& t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, X_s)X_s - \sigma(s, S_0)\tilde{X}_s|^p] \, ds \\
& \leq 3^{p-1} t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, X_s)X_s - \sigma(s, \tilde{X}_s)\tilde{X}_s|^p] \, ds \\
& + 3^{p-1} t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, \tilde{X}_s)\tilde{X}_s - \sigma(s, \hat{X}_s)\tilde{X}_s|^p] \, ds \\
& + 3^{p-1} t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, \hat{X}_s)\tilde{X}_s - \sigma(s, S_0)\tilde{X}_s|^p] \, ds \, .
\end{aligned}
$$

First, observe under Assumption 1 that if $t \leq 1$,

$$
t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, X_s)X_s - \sigma(s, \tilde{X}_s)\tilde{X}_s|^p] \, ds \leq \alpha^p \int_0^t \mathbb{E}^{\mathbb{Q}}[|X_s - \tilde{X}_s|^p] \, ds \, .
$$

Second, by Assumption 1 and the Hölder inequality with $t \leq 1$,

$$
\begin{aligned}
& t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s, \tilde{X}_s)\tilde{X}_s - \sigma(s, \hat{X}_s)\tilde{X}_s|^p] \, ds \\
& \leq t^{\frac{p}{2}-1} \int_0^t (\mathbb{E}^{\mathbb{Q}}[|\sigma(s, \tilde{X}_s) - \sigma(s, \hat{X}_s)|^{2p}])^{\frac{1}{2}} (\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s|^{2p}])^{\frac{1}{2}} \, ds \\
& \leq t^{\frac{p}{2}-1} \alpha^p \int_0^t (\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s - \hat{X}_s|^{2p}])^{\frac{1}{2}} {S_0}^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2} s} \, ds && \text{(A.1.3)} \\
& \leq \alpha^p (B_{2p})^{\frac{1}{2}} {S_0}^p t^{\frac{p}{2}-1} \int_0^t s^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2} s} \, ds \, . && \text{(A.1.4)} \\
& \leq \alpha^p (B_{2p})^{\frac{1}{2}} {S_0}^p t^p \int_0^1 e^{\frac{p(2p-1)\overline{\sigma}^2}{2} s} \, ds \, . && \text{(A.1.5)}
\end{aligned}
$$

Observe that Eq.(A.1.3) comes from direct calculation since $\tilde{X}_s$ is a log-normal random variable for each $s > 0$. The second inequality of Eq.(2.0.5) gives Eq.(A.1.4) whereas Eq.(A.1.5) holds under $t \leq 1$ with $p \geq 2$. Third, by Assumption 1 and the Hölder inequality with $t \le 1$,

$$\begin{aligned}
& t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(s,\hat{X}_s)\tilde{X}_s - \sigma(s,S_0)\tilde{X}_s|^p]\,ds \\
& \le t^{\frac{p}{2}-1}\int_0^t (\mathbb{E}^{\mathbb{Q}}[|\sigma(s,\hat{X}_s) - \sigma(s,S_0)|^{2p}])^{\frac{1}{2}}(\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_s|^{2p}])^{\frac{1}{2}}\,ds \\
& \le t^{\frac{p}{2}-1}\alpha^p\int_0^t (\mathbb{E}^{\mathbb{Q}}[|\hat{X}_s - S_0|^{2p}])^{\frac{1}{2}}{S_0}^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds \\
& \le \alpha^p\overline{\sigma}^p{S_0}^{2p}(M(2p))^{\frac{1}{2}}t^{\frac{p}{2}-1}\int_0^t s^{\frac{p}{2}}e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds \qquad \text{(A.1.6)} \\
& \le \alpha^p\overline{\sigma}^p{S_0}^{2p}(M(2p))^{\frac{1}{2}}\frac{1}{\frac{p}{2}+1}t^p e^{\frac{p(2p-1)\overline{\sigma}^2}{2}\vee 0} \qquad \text{(A.1.7)}
\end{aligned}$$

where $M(p)$ is defined as $\mathbb{E}^{\mathbb{Q}}[|Z|^p]$ while Z denotes a standard normal distribution. Since $\hat{X}_s$ is a normal random variable, it is easy to verify the inequality Eq.(A.1.6). From $t \le 1$, Eq.(A.1.7) follows. By combining the three inequalities, for $t \le 1$, we obtain

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] \\
& \le 3^{p-1}C_p\alpha^p\int_0^t \mathbb{E}^{\mathbb{Q}}[|X_s - \tilde{X}_s|^p]\,ds \\
& + \Big(3^{p-1}C_p\alpha^p(B_{2p})^{\frac{1}{2}}{S_0}^p\int_0^1 e^{\frac{p(2p-1)\overline{\sigma}^2}{2}s}\,ds \\
& + 3^{p-1}C_p\alpha^p\overline{\sigma}^p{S_0}^{2p}(M(2p))^{\frac{1}{2}}\frac{1}{\frac{p}{2}+1}e^{\frac{p(2p-1)\overline{\sigma}^2}{2}\vee 0}\Big)t^p\,.
\end{aligned}$$

From straightforward use of the Gronwall inequality, we can find a positive constant B'_p such that $\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^p] \le B'_p t^p$. Without loss of generality, we can identify B'_p with B_p or otherwise just take a bigger one.

The proof for the second inequality of Eq.(2.0.6) is nearly a repetition of the proof for the second inequality of Eq.(2.0.5); hence, it is omitted. Here, we examine only the first inequality of Eq.(2.0.6). Again, restrict $p \ge 2$.

By the Burkholder–Davis–Gundy inequality and the Jensen inequality,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^p] &\leq C_p \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^t (\nu(s, X_s)Y_s - \nu(s, S_0)\tilde{Y}_s)^2\, ds\right)^{\frac{p}{2}}\right] \\
&\leq 3^{p-1} C_p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\nu(s, X_s)Y_s - \nu(s, X_s)\tilde{Y}_s|^p]\, ds \\
&+ 3^{p-1} C_p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\nu(s, X_s)\tilde{Y}_s - \nu(s, \hat{X}_s)\tilde{Y}_s|^p]\, ds \\
&+ 3^{p-1} C_p t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\nu(s, \hat{X}_s)\tilde{Y}_s - \nu(s, S_0)\tilde{Y}_s|^p]\, ds
\end{aligned}
$$

for some constant $C_p > 0$. The remainder of the proof is similar to the proof for Eq.(2.0.5). First,

$$
t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\nu(s, X_s)Y_s - \nu(s, X_s)\tilde{Y}_s|^p]\, ds \leq \alpha^p \int_0^t \mathbb{E}^{\mathbb{Q}}[|Y_s - \tilde{Y}_s|^p]\, ds
$$

for $t \leq 1$. For this inequality, we use $|\nu(t, x)| \leq \alpha$ for all $t > 0$ and $x \in \mathbb{R}$, which comes from Assumption 1. Second, observe from Assumption 1, the Hölder inequality, Eq.(2.0.5), and direct computation with regard to a log-normal random variable $\tilde{Y}_s$ that

$$
\begin{aligned}
& t^{\frac{p}{2}-1} \int_0^t \mathbb{E}^{\mathbb{Q}}[|\nu(s, X_s)\tilde{Y}_s - \nu(s, \hat{X}_s)\tilde{Y}_s|^p]\, ds \\
&\leq \alpha^p t^{\frac{p}{2}-1} \int_0^t (\mathbb{E}^{\mathbb{Q}}[|X_s - \hat{X}_s|^{2p}])^{\frac{1}{2}} (\mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_s|^{2p}])^{\frac{1}{2}}\, ds \\
&\leq \alpha^p 2^p (B_{2p})^{\frac{1}{2}} t^{\frac{p}{2}-1} \int_0^t s^p e^{\frac{p(2p-1)\alpha^2}{2} s}\, ds \\
&\leq \alpha^p 2^p (B_{2p})^{\frac{1}{2}} t^p \int_0^1 e^{\frac{p(2p-1)\alpha^2}{2} s}\, ds \qquad \text{(A.1.8)}
\end{aligned}
$$

where Eq.(A.1.8) holds under $t \leq 1$. Third, use Assumption 1, the Hölder inequality, and direct computation with regard to a normal random variable $\hat{X}_s$ and a log-normal random variable $\tilde{Y}_s$ to obtain

$$\begin{aligned}
&t^{\frac{p}{2}-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|\nu(s,\hat{X}_s)\tilde{Y}_s - \nu(s,S_0)\tilde{Y}_s|^p]\,ds \\
&\leq \alpha^p t^{\frac{p}{2}-1}\int_0^t (\mathbb{E}^{\mathbb{Q}}[|\hat{X}_s - S_0|^{2p}])^{\frac{1}{2}}(\mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_s|^{2p}])^{\frac{1}{2}}\,ds \\
&\leq \alpha^p \overline{\sigma}^p {S_0}^p (M(2p))^{\frac{1}{2}} t^{\frac{p}{2}-1}\int_0^t s^{\frac{p}{2}} e^{\frac{p(2p-1)\alpha^2}{2}s}\,ds \\
&\leq \alpha^p \overline{\sigma}^p {S_0}^p (M(2p))^{\frac{1}{2}} \frac{1}{\frac{p}{2}+1} t^p e^{\frac{p(2p-1)\alpha^2}{2}\vee 0}. \qquad \text{(A.1.9)}
\end{aligned}$$

The last inequality Eq.(A.1.9) holds under $t \leq 1$. Combine the three inequalities to get

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^p] \\
&\leq 3^{p-1}C_p\alpha^p \int_0^t \mathbb{E}^{\mathbb{Q}}[|Y_s - \tilde{Y}_s|^p]\,ds \\
&+ \Big(3^{p-1}C_p\alpha^p 2^p (B_{2p})^{\frac{1}{2}}\int_0^1 e^{\frac{p(2p-1)\alpha^2}{2}s}\,ds \\
&+ 3^{p-1}C_p\alpha^p\overline{\sigma}^p {S_0}^p (M(2p))^{\frac{1}{2}}\frac{1}{\frac{p}{2}+1} e^{\frac{p(2p-1)\alpha^2}{2}\vee 0}\Big) t^p\,.
\end{aligned}$$

Apply the Gronwall inequality to get the bound B_p. □

A.2 Proof for Lemma 2.0.2

Before we prove Lemma 2.0.2, we state and prove the generalized version of it. Later, we will show that the following lemma is actually a sufficient condition.

Lemma A.2.1. *Given a measure space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a Brownian motion $(W_t)_{t\geq 0}$, suppose that a process $(\theta_t)_{t\geq 0}$ is adapted to the Brownian filtration $(\mathcal{F}_t^W)_{t\geq 0}$ and is uniformly bounded. More precisely, there is a constant*

$C > 0$ such that $|\theta_t(\omega)| \le C$ for any $t \ge 0$ and $\omega \in \Omega$. Define a continuous martingale process $(M_t)_{t\ge 0}$ as

$$M_t := M_0 e^{-\frac{1}{2}\int_0^t \theta_s^2\,ds + \int_0^t \theta_s\,dW_s}\,, \quad M_0 > 0\,.$$

Then, for any $\xi \in \mathbb{R}$, the following three statements hold.

(i) $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}[M_T^{\xi}] = M_0^{\xi}\,.$

(ii) $\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le t\le T} M_t^{\xi}\Big] < \infty$ *, for any $T > 0$. Also,* $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le t\le T} M_t^{\xi}\Big] = M_0^{\xi}\,.$

(iii) $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T M_t\,dt\right)^{\xi}\right] = M_0^{\xi}\,.$

Proof. Observe that

$$M_0^{\xi} e^{k(\xi)T} e^{-\frac{1}{2}\int_0^T \xi^2\theta_s^2\,ds + \int_0^T \xi\theta_s\,dW_s} \le M_T^{\xi} \le M_0^{\xi} e^{K(\xi)T} e^{-\frac{1}{2}\int_0^T \xi^2\theta_s^2\,ds + \int_0^T \xi\theta_s\,dW_s}\,, \tag{A.2.1}$$

where $k(\xi) := \frac{\xi(\xi-1)}{2} \wedge 0$, $K(\xi) := \frac{\xi(\xi-1)}{2} \vee 0$. By taking the expectation $\mathbb{E}^{\mathbb{Q}}$ on both sides, we obtain $M_0^{\xi} e^{k(\xi)T} \le \mathbb{E}^{\mathbb{Q}}[M_T^{\xi}] \le M_0^{\xi} e^{K(\xi)T}$ because $\big(e^{-\frac{1}{2}\int_0^t \xi^2\theta_s^2\,ds + \int_0^t \xi\theta_s\,dW_s}\big)_{t\ge 0}$ is a continuous martingale. Take $T \to 0$ to get the limit $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}[M_T^{\xi}] = M_0^{\xi}\,.$

Suppose that $\xi > 0\,.$ Choose p such that $p \ge 1$ and $\xi p > 1$. Use the Jensen inequality and the Doob L^p inequality to obtain

$$\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\max_{0\le t\le T} M_t\right)^{\xi}\right]\right)^p \le \mathbb{E}^{\mathbb{Q}}\left[\left(\max_{0\le t\le T} M_t\right)^{\xi p}\right] \le \left(\frac{\xi p}{\xi p - 1}\right)^{\xi p} \mathbb{E}^{\mathbb{Q}}[M_T^{\xi p}]\,.$$

Since $\mathbb{E}^{\mathbb{Q}}[M_T^{\xi p}] \le M_0^{\xi p} e^{K(\xi p)T} < \infty$, we get that $\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le t\le T} M_t^{\xi}\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\Big(\max_{0\le t\le T} M_t\Big)^{\xi}\Big]$ $< \infty$ for any $T > 0\,.$ Note that $\max_{0\le t\le T} M_t^{\xi} \searrow M_0^{\xi}$ almost surely as $T \to 0$. Thus, from the Lebesgue dominated convergence theorem, $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le t\le T} M_t^{\xi}\Big] =$

M_0^ξ . Next, suppose that $\xi < 0$. Observe that

$$\frac{1}{M_t} = \frac{1}{M_0} e^{\frac{1}{2}\int_0^t \theta_s^2\, ds - \int_0^t \theta_s\, dW_s} = \frac{1}{M_0} e^{\int_0^t \theta_s^2\, ds} e^{-\frac{1}{2}\int_0^t \theta_s^2\, ds - \int_0^t \theta_s\, dW_s} \le \frac{1}{M_0} e^{C^2 t} N_t\,,$$

where $N_t := e^{-\frac{1}{2}\int_0^t (-\theta_s)^2\, ds + \int_0^t -\theta_s\, dW_s}$. Take the $(-\xi)$th power to the above-mentioned inequality to obtain

$$\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} M_t^\xi\right] \le M_0^\xi e^{-\xi C^2 T}\, \mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} N_t^{-\xi}\right].$$

Observe from the proof for the case $\xi > 0$ that we have already proved $\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} N_t^{-\xi}\right] < \infty$ for any $T > 0$. Therefore, $\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} M_t^\xi\right] < \infty$ for any $T > 0$. By the Lebesgue dominated convergence theorem, $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} M_t^\xi\right] = M_0^\xi$ follows.

Finally, we prove that $\lim_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T M_t\, dt\right)^\xi\right] = M_0^\xi$. From the Fatou lemma, it suffices to show that $\limsup_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T M_t\, dt\right)^\xi\right] \le M_0^\xi$. This is straightforward from the inequality $\left(\frac{1}{T}\int_0^T M_t\, dt\right)^\xi \le \max_{0\le t\le T} M_t^\xi$ for $\xi \in \mathbb{R}$. □

Proof of Lemma 2.0.2. We now prove Lemma 2.0.2. Observe that the processes $(X_t)_{t\ge 0}$, $(\tilde{X}_t)_{t\ge 0}$, $(Y_t)_{t\ge 0}$, $(\tilde{Y}_t)_{t\ge 0}$ all satisfy the condition in Lemma A.2.1. Thus, from Lemma A.2.1 and the Hölder inequality

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\right] \\
&\le \mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} X_t^{p_1} \max_{0\le t\le T} \tilde{X}_t^{p_2} \max_{0\le t\le T} Y_t^{p_3} \max_{0\le t\le T} \tilde{Y}_t^{p_4}\right] \\
&\le \left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} X_t^{4p_1}\right]\right)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} \tilde{X}_t^{4p_2}\right]\right)^{\frac{1}{4}}
\end{aligned}$$

$$\times \left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} Y_t^{4p_3}\right]\right)^{\frac{1}{4}} \left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} \tilde{Y}_t^{4p_4}\right]\right)^{\frac{1}{4}} ,$$

we obtain $\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\right] < \infty$ for any $T > 0$. Moreover, it is followed by the upper bound $\limsup_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\right] \le S_0^{p_1+p_2}$. The Fatou lemma yields the lower bound $S_0^{p_1+p_2} \le \liminf_{T\to 0} \mathbb{E}^{\mathbb{Q}}\left[\max_{0\le t\le T} Z_t^{p_1,p_2,p_3,p_4}\right]$.

The inequality Eq.(2.0.7) can be proved similarly. From the Fatou lemma, it is sufficient to show only the upper bound

$$\limsup_{T\to 0} \mathbb{E}^{\mathbb{Q}}\Bigg[Z_T^{q_1,q_2,q_3,q_4} \left(\frac{1}{T}\int_0^T X_t\, dt\right)^{q_5} \left(\frac{1}{T}\int_0^T \tilde{X}_t\, dt\right)^{q_6}$$
$$\left(\frac{1}{T}\int_0^T Y_t\, dt\right)^{q_7} \left(\frac{1}{T}\int_0^T \tilde{Y}_t\, dt\right)^{q_8}\Bigg] \le S_0^{q_1+q_2+q_5+q_6} .$$

Meanwhile, the upper bound is implied from the use of the Hölder inequality

$$\mathbb{E}^{\mathbb{Q}}\left[Z_T^{q_1,q_2,q_3,q_4} \left(\frac{1}{T}\int_0^T X_t\, dt\right)^{q_5} \left(\frac{1}{T}\int_0^T \tilde{X}_t\, dt\right)^{q_6} \left(\frac{1}{T}\int_0^T Y_t\, dt\right)^{q_7} \left(\frac{1}{T}\int_0^T \tilde{Y}_t\, dt\right)^{q_8}\right]$$

$$\le \left(\mathbb{E}^{\mathbb{Q}}\left[Z_T^{5q_1,5q_2,5q_3,5q_4}\right]\right)^{\frac{1}{5}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T X_t\, dt\right)^{5q_5}\right]\right)^{\frac{1}{5}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{X}_t\, dt\right)^{5q_6}\right]\right)^{\frac{1}{5}}$$

$$\times \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T Y_t\, dt\right)^{5q_7}\right]\right)^{\frac{1}{5}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{Y}_t\, dt\right)^{5q_8}\right]\right)^{\frac{1}{5}} ,$$

with Lemma A.2.1. Hence, we get the desired result. □

Appendix B

Detailed proof of Chapter 3

B.1 Proof for Lemma 3.0.1

Proof. Define $\tilde{S}_t := e^{-(r-q)t} S_t$. Then, by Ito calculus, we get

$$d\tilde{S}_t = \sigma(t, e^{(r-q)t}\tilde{S}_t)\tilde{S}_t \, dW_t \, , \quad \tilde{S}_0 = S_0 \, .$$

Using α in Assumption 1 and the inequality $(x+y)^2 \leq 2x^2 + 2y^2$ for $x, y \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[|\tilde{S}_t - X_t|^2] \\
&= \int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(u, e^{(r-q)u}\tilde{S}_u)\tilde{S}_u - \sigma(u, X_u)X_u|^2] \, du \\
&\leq 2\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(u, e^{(r-q)u}\tilde{S}_u)\tilde{S}_u - \sigma(u, \tilde{S}_u)\tilde{S}_u|^2] \, du \\
&+ 2\int_0^t \mathbb{E}^{\mathbb{Q}}[|\sigma(u, \tilde{S}_u)\tilde{S}_u - \sigma(u, X_u)X_u|^2] \, du \\
&\leq 2\alpha^2 \int_0^t |e^{(r-q)u} - 1|^2 \mathbb{E}^{\mathbb{Q}}[\tilde{S}_u^4] \, du + 2\alpha^2 \int_0^t \mathbb{E}^{\mathbb{Q}}[|\tilde{S}_u - X_u|^2] \, du \, .
\end{aligned}$$

Since $\mathbb{E}^{\mathbb{Q}}[\tilde{S}_u^4] \le S_0^4 e^{6\overline{\sigma}^2 u}$ holds for any $u \ge 0$,

$$2\alpha^2 \int_0^t |e^{(r-q)u} - 1|^2 \mathbb{E}^{\mathbb{Q}}[\tilde{S}_u^4]\, du = \mathcal{O}(t^3)$$

as $t \to 0$. Therefore, by the Gronwall inequality, we get $\mathbb{E}^{\mathbb{Q}}[|\tilde{S}_t - X_t|^2] = \mathcal{O}(t^3)$ as $t \to 0$. Further, note that from $(x+y)^2 \le 2x^2 + 2y^2$ for $x, y \ge 0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^2] &\le 2\mathbb{E}^{\mathbb{Q}}[|S_t - \tilde{S}_t|^2] + 2\mathbb{E}^{\mathbb{Q}}[|\tilde{S}_t - X_t|^2] \\
&\le 2|e^{(r-q)t} - 1|^2 \mathbb{E}^{\mathbb{Q}}[\tilde{S}_t^2] + 2\mathbb{E}^{\mathbb{Q}}[|\tilde{S}_t - X_t|^2].
\end{aligned}$$

Hereafter, from $\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t^2] \le S_0^2 e^{\overline{\sigma}^2 t}$, it is implied that $\mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^2] = \mathcal{O}(t^2)$. Further, note that $\mathbb{E}^{\mathbb{Q}}[|S_t - X_t|] \le (\mathbb{E}^{\mathbb{Q}}[|S_t - X_t|^2])^{\frac{1}{2}} = \mathcal{O}(t)$. Thus, from the inequalities

$$\left| P_A(T) - e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T} \int_0^T X_t\, dt \right) \right] \right| \le \beta e^{-rT}\, \frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}}[|S_t - X_t|]\, dt = \mathcal{O}(T)\,,$$

$$\left| P_E(T) - e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[\Phi(X_T)] \right| \le \beta e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[|S_T - X_T|] = \mathcal{O}(T)\,,$$

we obtain the desired proof. $\square$

Appendix C

Detailed proof of Chapter 4

C.1 Proof for Lemma 4.0.1

By the Lebesgue dominated convergence theorem, the Asian and European delta values can be expressed in terms of the derivative $\Phi^{'}$ of the payoff function.

Lemma C.1.1. *For the process S stated in Eq.*(2.0.1) *under Assumptions 1 and 3, we have*

$$\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T S_t\,dt\right)\right] = \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi^{'}\left(\frac{1}{T}\int_0^T S_t\,dt\right)\frac{1}{T}\int_0^T S_t\,dt\right],$$
$$\frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] = \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}[\Phi^{'}(S_T)S_T].$$

Here, the derivative $\Phi^{'}$ is defined almost everywhere with respect to the Lebesgue measure.

Proof. By the Lebesgue dominated convergence theorem, it suffices to show that the distribution measures of S_T and $\frac{1}{T}\int_0^T S_t\,dt$ are both absolutely continuous with respect to the Lebesgue measure for any $T \geq 0$. From Theorem 2.3.1 of Nualart (1995), a density of S_T always exists. Observe from Assumption 1 that the first variation process of S is the unique solution of the stochastic differential equation

$$dZ_t = (r-q)Z_t\,dt + \nu(t,S_t)Z_t\,dW_t\,, \quad Z_0 = 1 \tag{C.1.1}$$

and that

$$D_u S_t = \frac{Z_t}{Z_u}\sigma(u,S_u)S_u\mathbb{1}_{\{u\le t\}}\,.$$

Therefore, from Proposition 7.1.2 of Nualart and Nualart (2018), it is easy to check that a density of $\frac{1}{T}\int_0^T S_t\,dt$ always exists. This completes the proof. $\square$

Proof of Lemma 4.0.1. Define a process $\theta_t := \frac{r-q}{\sigma(t,S_t)}, t\ge 0$. Then, the process

$$M_t := e^{-\frac{1}{2}\int_0^t \theta_u^2\,du - \int_0^t \theta_u\,dW_u}$$

is a continuous martingale since θ is a bounded process. By the Girsanov theorem, $dB_t := dW_t + \theta_t\,dt$, $0\le t\le T$ is a Brownian motion under the measure $d\mathbb{P} := M_T\,d\mathbb{Q}$ on $\mathcal{F}_T^W$. Since $dS_t = \sigma(t,S_t)S_t\,dB_t$ under the measure $\mathbb{P}$, Lemma C.1.1 states that

$$\begin{aligned}
\Delta_A(T) &= \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi'\left(\frac{1}{T}\int_0^T S_t\,dt\right)\frac{1}{T}\int_0^T S_t\,dt\right]\\
&= \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{P}}\left[\Phi'\left(\frac{1}{T}\int_0^T S_t\,dt\right)\frac{1}{T}\int_0^T S_t\,dt\,\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\\
&= \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi'\left(\frac{1}{T}\int_0^T X_t\,dt\right)\frac{1}{T}\int_0^T X_t\,dt\,e^{-\frac{1}{2}\int_0^T \eta_t^2\,dt+\int_0^T \eta_t\,dW_t}\right],
\end{aligned}$$

where $\eta_t = \frac{r-q}{\sigma(t,X_t)}$. Therefore, by the Cauchy–Schwarz inequality and the inequality $|\Phi'|\le\beta$ from Assumption 3,

$$\left|\Delta_A(T) - \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\Phi'\left(\frac{1}{T}\int_0^T X_t\,dt\right)\frac{1}{T}\int_0^T X_t\,dt\right]\right|$$

$$\begin{aligned}
&\leq \beta \frac{e^{-rT}}{S_0} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{T} \int_0^T X_t \, dt \left| e^{-\frac{1}{2}\int_0^T \eta_t^2 \, dt + \int_0^T \eta_t \, dW_t} - 1 \right| \right] \\
&\leq \beta \frac{e^{-rT}}{S_0} \left(\frac{1}{T} \int_0^T \mathbb{E}^{\mathbb{Q}}[X_t^2] \, dt \right)^{\frac{1}{2}} \left(e^{\frac{(r-q)^2}{\underline{\sigma}^2}T} - 1 \right)^{\frac{1}{2}} .
\end{aligned}$$

Since $\mathbb{E}^{\mathbb{Q}}[X_t^2] \leq S_0^2 e^{\overline{\sigma}^2 t}$ and $e^{\frac{(r-q)^2}{\underline{\sigma}^2}T} - 1 = \mathcal{O}(T)$, the proof for the Asian delta value is complete. The proof for the European delta value can be obtained similarly. □

C.2 Proof for Lemma 4.1.1

Proof. We prove the first inequality of Eq.(4.1.4). It is sufficient to show this for $p \geq 1$. By Assumption 1 and the Jensen inequality $(\frac{x+y}{2})^p \leq \frac{x^p+y^p}{2}$ for $x, y \geq 0$,

$$\begin{aligned}
&|u_t - \tilde{u}_t|^p \\
&= \left| \frac{2(Y_t + \tilde{Y}_t)(Y_t - \tilde{Y}_t)}{\sigma(t, X_t)X_t} + \frac{2\tilde{Y}_t^2(\sigma(t, \tilde{X}_t)\tilde{X}_t - \sigma(t, X_t)X_t)}{\sigma(t, X_t)X_t\sigma(t, \tilde{X}_t)\tilde{X}_t} \right|^p \\
&\leq 2^{2p-1} \left| \frac{(Y_t + \tilde{Y}_t)(Y_t - \tilde{Y}_t)}{\sigma(t, X_t)X_t} \right|^p + 2^{2p-1} \left| \frac{\tilde{Y}_t^2(\sigma(t, \tilde{X}_t)\tilde{X}_t - \sigma(t, X_t)X_t)}{\sigma(t, X_t)X_t\sigma(t, \tilde{X}_t)\tilde{X}_t} \right|^p \\
&\leq \frac{2^{2p-1}}{\underline{\sigma}^p} \left| \frac{(Y_t + \tilde{Y}_t)(Y_t - \tilde{Y}_t)}{X_t} \right|^p + \frac{2^{2p-1}\alpha^p}{\underline{\sigma}^{2p}} \left(\frac{\tilde{Y}_t^{2p}}{{X_t}^p \tilde{X}_t^p} \right) |X_t - \tilde{X}_t|^p .
\end{aligned}$$

Observe from the Jensen inequality $(\frac{x+y}{2})^p \leq \frac{x^p+y^p}{2}$ for $x, y \geq 0$ that

$$\begin{aligned}
\left| \frac{(Y_t + \tilde{Y}_t)(Y_t - \tilde{Y}_t)}{X_t} \right|^p &= \left| \frac{Y_t(Y_t - \tilde{Y}_t)}{X_t} + \frac{\tilde{Y}_t(Y_t - \tilde{Y}_t)}{X_t} \right|^p \\
&\leq 2^{p-1} Y_t^p {X_t}^{-p} |Y_t - \tilde{Y}_t|^p + 2^{p-1} \tilde{Y}_t^p {X_t}^{-p} |Y_t - \tilde{Y}_t|^p .
\end{aligned}$$

Note from the Hölder inequality and Lemma 2.0.1 that for $0 \leq t \leq 1$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p X_t^{-p}|Y_t - \tilde{Y}_t|^p\right] &\leq \left(\mathbb{E}^{\mathbb{Q}}\left[Y_t^{2p} X_t^{-2p}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^{2p}]\right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\left(Y_s^{2p} X_s^{-2p}\right)\Big]\right)^{\frac{1}{2}} (B_{2p})^{\frac{1}{2}} t^p ,\end{aligned}$$

with some positive constant B_{2p}. By the same argument,

$$\mathbb{E}^{\mathbb{Q}}\left[\tilde{Y}_t^p X_t{}^{-p}|Y_t - \tilde{Y}_t|^p\right] \leq \left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\left(\tilde{Y}_s^{2p} X_s^{-2p}\right)\Big]\right)^{\frac{1}{2}} (B_{2p})^{\frac{1}{2}} t^p$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\tilde{Y}_t^{2p} X_t{}^{-p}\tilde{X}_t^{-p}|X_t - \tilde{X}_t|^p\right] \leq \left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\left(\tilde{Y}_s^{4p} X_s^{-2p}\tilde{X}_s^{-2p}\right)\Big]\right)^{\frac{1}{2}} (B_{2p})^{\frac{1}{2}} t^p .$$

Therefore, for $0 \leq t \leq 1$, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|u_t - \tilde{u}_t|^p] \leq{}& \frac{2^{3p-2}}{\underline{\sigma}^p}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p X_t^{-p}|Y_t - \tilde{Y}_t|^p\right] + \frac{2^{3p-2}}{\underline{\sigma}^p}\mathbb{E}^{\mathbb{Q}}\left[\tilde{Y}_t^p X_t{}^{-p}|Y_t - \tilde{Y}_t|^p\right] \\ &+ \frac{2^{2p-1}\alpha^p}{\underline{\sigma}^{2p}}\mathbb{E}^{\mathbb{Q}}\left[\tilde{Y}_t^{2p} X_t{}^{-p}\tilde{X}_t^{-p}|X_t - \tilde{X}_t|^p\right] \\ \leq{}& D_p t^p ,\end{aligned}$$

where a constant $D_p > 0$ is taken as

$$\begin{aligned}D_p :={}& \frac{2^{3p-2}}{\underline{\sigma}^p}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\left(Y_s^{2p} X_s^{-2p}\right)\Big]\right)^{\frac{1}{2}} (B_{2p})^{\frac{1}{2}} \\ &+ \frac{2^{3p-2}}{\underline{\sigma}^p}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\left(\tilde{Y}_s^{2p} X_s^{-2p}\right)\Big]\right)^{\frac{1}{2}} (B_{2p})^{\frac{1}{2}}\end{aligned}$$

$$+\frac{2^{2p-1}\alpha^p}{\underline{\sigma}^{2p}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le s\le 1}\left(\tilde{Y}_s^{4p}X_s^{-2p}\tilde{X}_s^{-2p}\right)\Big]\right)^{\frac{1}{2}}(B_{2p})^{\frac{1}{2}}\,.$$

From Lemma 2.0.2, $D_p<\infty$; hence, it is well defined.

The second inequality of Eq.(4.1.4) can be obtained similarly. The only difference is that we have to handle an indicator function that appears in the definition of $\hat{u}_s$. We may assume that $p\ge 1$. Observe from the Jensen inequality that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|\tilde{u}_t-\hat{u}_t|^p]\le{}& 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t}\right|^p\mathbb{1}_{\{\hat{X}_t<\frac{S_0}{2}\}}\right]\\&+2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t}-\frac{2\hat{Y}_t^2}{\sigma(t,\hat{X}_t)\hat{X}_t}\right|^p\mathbb{1}_{\{\hat{X}_t\ge\frac{S_0}{2}\}}\right].\end{aligned}$$

Following the arguments used to prove the first inequality of Eq.(4.1.4), we can show that there exists some positive constant D_p such that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t}-\frac{2\hat{Y}_t^2}{\sigma(t,\hat{X}_t)\hat{X}_t}\right|^p\mathbb{1}_{\{\hat{X}_t\ge\frac{S_0}{2}\}}\right]\le D_pt^p$$

for small $0\le t\le 1$. Now, note from Assumption 1 and the Hölder inequality that

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\tilde{Y}_t^2}{\sigma(t,\tilde{X}_t)\tilde{X}_t}\right|^p\mathbb{1}_{\{\hat{X}_t<\frac{S_0}{2}\}}\right]\\&\le\frac{2^p}{\underline{\sigma}^p}\left(\mathbb{E}^{\mathbb{Q}}\left[\tilde{Y}_t^{4p}\tilde{X}_t^{-2p}\right]\right)^{\frac{1}{2}}\left(\mathbb{Q}\left\{\hat{X}_t<\frac{S_0}{2}\right\}\right)^{\frac{1}{2}}\\&\le\frac{2^p}{\underline{\sigma}^p}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\le s\le 1}\left(\tilde{Y}_s^{4p}\tilde{X}_s^{-2p}\right)\Big]\right)^{\frac{1}{2}}\left(\mathbb{Q}\left\{\hat{X}_t<\frac{S_0}{2}\right\}\right)^{\frac{1}{2}}\,.\end{aligned}$$

Let Z be the standard normal variable with respect to the measure $\mathbb{Q}$ and $N(\cdot)$ be a cumulative function of Z. To prove the desired inequality, we rely on the property of a cumulative function $N(\cdot)$. Since $\hat{X}_t$ is a normal random variable, it is implied from Assumption 1 that the following inequality holds for any $t > 0$.

$$\mathbb{Q}\left\{\hat{X}_t < \frac{S_0}{2}\right\} = N\left(-\frac{1}{2\sqrt{\int_0^t \sigma(u,S_0)^2\,du}}\right) \le N\left(-\frac{1}{2\overline{\sigma}\sqrt{t}}\right). \quad \text{(C.2.1)}$$

However, it is easy to check from the definition of $N(\cdot)$ that $N\left(-\frac{1}{2\overline{\sigma}\sqrt{t}}\right)$ $= o(t^q)$ for any $q > 0$ as $t \to 0$. Furthermore, from Lemma 2.0.2, $\mathbb{E}^{\mathbb{Q}}\left[\max\limits_{0\le s\le 1}\left(\tilde{Y}_s^{4p}\tilde{X}_s^{-2p}\right)\right] < \infty$. This completes the proof of Eq.(4.1.4).

The proof for Eq.(4.1.5) is much simpler with many key ideas shared with the proof of Eq.(4.1.4). Assume that $p \ge 1$. Then, by the Jensen inequality and the Hölder inequality,

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}[|TF - T\tilde{F}|^p] \\ &\le \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^{2p}]\,dt\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^{-2p}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-2p}\right]\right)^{\frac{1}{2}}.\end{aligned}$$

Note from Lemma 2.0.1 that for $0 \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[|TF - T\tilde{F}|^p] \le \left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\le s\le 1} Y_s^{-2p}\max_{0\le s\le 1}\tilde{Y}_s^{-2p}\right]\right)^{\frac{1}{2}}\left(\frac{B_{2p}}{2p+1}\right)^{\frac{1}{2}} T^p.$$

From Lemma 2.0.2 and the Cauchy–Schwarz inequality, $\mathbb{E}^{\mathbb{Q}}\left[\max\limits_{0\le s\le 1} Y_s^{-2p}\max\limits_{0\le s\le 1}\tilde{Y}_s^{-2p}\right]$ $< \infty$. Here, we prove the first inequality of Eq.(4.1.5).

For the second inequality of Eq.(4.1.5), we may also assume that $p \ge 1$. Note from the Jensen inequality $(\frac{x+y}{2})^p \le \frac{x^p+y^p}{2}$, $x, y > 0$, that

$$\mathbb{E}^{\mathbb{Q}}[|T\tilde{F} - T\hat{F}|^p]$$

$$\leq 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt<\frac{1}{2}\}}\right]$$
$$+\,2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |\tilde{Y}_t-\hat{Y}_t|\,dt\right)^{p}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^{-p}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt\geq\frac{1}{2}\}}\right].$$

Observe from the Jensen inequality, the Hölder inequality, and Lemma 2.0.1 that for $0\leq T\leq 1$,

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |\tilde{Y}_t-\hat{Y}_t|\,dt\right)^{p}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^{-p}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt\geq\frac{1}{2}\}}\right]$$

$$\leq 2^{p}\,\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |\tilde{Y}_t-\hat{Y}_t|\,dt\right)^{p}\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\right]$$
$$\leq 2^{p}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_t-\hat{Y}_t|^{2p}\,dt]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\tilde{Y}_s^{-2p}\Big]\right)^{\frac{1}{2}}\leq D_pT^p,$$

where $D_p := 2^p\left(\frac{B_{2p}}{2p+1}\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\tilde{Y}_s^{-2p}\Big]\right)^{\frac{1}{2}}$. From Lemma 2.0.2, $D_p<\infty$. Next, observe from the Fubini theorem with regard to a stochastic integral and Assumption 1 that

$$\begin{aligned}\mathbb{Q}\left\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt<\frac{1}{2}\right\} &= \mathbb{Q}\left\{1+\frac{1}{T}\int_0^T\int_0^T \nu(s,S_0)\mathbb{1}_{\{s\leq t\}}\,dW_s\,dt<\frac{1}{2}\right\}\\ &= \mathbb{Q}\left\{\frac{1}{T}\int_0^T \nu(s,S_0)(T-s)\,dW_s<-\frac{1}{2}\right\}\\ &= \mathbb{Q}\left\{\frac{1}{T}\sqrt{\int_0^T \nu^2(s,S_0)(T-s)^2\,ds}\,Z<-\frac{1}{2}\right\}\\ &\leq N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right),\end{aligned} \tag{C.2.2}$$

where Z denotes a standard normal random variable and $N(\cdot)$ denotes a cumulative function of Z . Hence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-p}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt<\frac{1}{2}\}}\right] \\
&\leq \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^{-2p}\right]\right)^{\frac{1}{2}}\left(N\Big(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\Big)\right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq s\leq 1}\tilde{Y}_s^{-2p}\Big]\right)^{\frac{1}{2}}\left(N\Big(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\Big)\right)^{\frac{1}{2}}
\end{aligned}$$

for $0 \leq T \leq 1$. By Lemma 2.0.2, $\mathbb{E}^{\mathbb{Q}}\left[\max_{0\leq s\leq 1}\tilde{Y}_s^{-2p}\right] < \infty$. Since $\left(N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right)\right)^{\frac{1}{2}} = o(T^q)$ for any $q > 0$ as $T \to 0$, the second inequality of (4.1.5) is proven, and so is Lemma 4.1.1. $\square$

C.3 Proof for Lemma 4.1.2

Proof. First, we present the proof for Eq.(4.1.8). In Benhamou (2000); Nualart (1995), D_sX_t is explicitly expressed as

$$D_sX_t = \frac{Y_t}{Y_s}\sigma(s, X_s)X_s\mathbb{1}_{\{s\leq t\}} \tag{C.3.1}$$

under Assumption 1. Therefore, for any $p > 0$ and $0 \leq t \leq 1$, by Assumption 1,

$$|D_sX_t|^p \leq \overline{\sigma}^p Y_t^p\Big[\max_{0\leq u\leq 1}\left(X_u^pY_u^{-p}\right)\Big]\mathbb{1}_{\{s\leq t\}}\ .$$

By taking the expectation $\mathbb{E}^{\mathbb{Q}}$ on both sides, from the Hölder inequality, we get

$$\sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}[|D_sX_t|^p] \leq \overline{\sigma}^p\left(\mathbb{E}^{\mathbb{Q}}[Y_t^{2p}]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\leq u\leq 1}\left(X_u^{2p}Y_u^{-2p}\right)\right]\right)^{\frac{1}{2}} .$$

From Lemma 2.0.2, the proof of Eq.(4.1.8) is complete.

Next, let us prove Eq.(4.1.9). The only nontrivial inequality of Eq.(4.1.9) is the first one. From Nualart (1995), $D_s\tilde{Y}_t$ and $D_s\hat{Y}_t$ can be computed as

$$D_s\tilde{Y}_t = \nu(s,S_0)\tilde{Y}_t\mathbb{1}_{\{s\leq t\}}\,, \quad D_s\hat{Y}_t = \nu(s,S_0)\mathbb{1}_{\{s\leq t\}}\,. \tag{C.3.2}$$

Therefore, from Assumption 1 and Lemma 2.0.2, it is easy to check that both $\sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}[|D_s\tilde{Y}_t|^p]$ and $\sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}[|D_s\hat{Y}_t|^p]$ are bounded by some constant $E_p>0$ in $0\leq t\leq 1$. To prove the first inequality of Eq.(4.1.9), use Malliavin calculus theory presented in Nualart (1995) to express D_sY_t by

$$\begin{aligned} D_sY_t = Y_t\bigg[\nu(s,X_s) &- \int_0^t \nu(u,X_u)\rho(u,X_u)D_sX_u\,du \\ &+ \int_0^t \rho(u,X_u)D_sX_u\,dW_u\bigg]\mathbb{1}_{\{s\leq t\}}\,. \end{aligned} \tag{C.3.3}$$

Now, take an absolute value on both sides of Eq.(C.3.3). Using the inequalities $|\nu|\leq\alpha$, $|\rho|\leq\alpha$, which follow from Assumption 1, observe that

$$|D_sY_t| \leq \left[\alpha Y_t + \alpha^2 Y_t\int_0^t |D_sX_u|\,du + Y_t\left|\int_0^t \rho(u,X_u)D_sX_u\,dW_u\right|\right]\mathbb{1}_{\{s\leq t\}}\,.$$

We may assume that $p\geq 1$. For any $s\geq 0$, by the Jensen inequality,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[|D_sY_t|^p] \leq 3^{p-1}\alpha^p\mathbb{E}^{\mathbb{Q}}[Y_t^p] &+ 3^{p-1}\alpha^{2p}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t |D_sX_u|^p\,du\right]t^{p-1} \\ &+ 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t \rho(u,X_u)D_sX_u\,dW_u\right|^p\right]. \end{aligned}$$

Note from the Jensen inequality, the Hölder inequality, and Eq.(4.1.8) that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[Y_t^p \int_0^t |D_s X_u|^p \, du\right] t^{p-1} \\
&\leq \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}} \left(\int_0^t \mathbb{E}^{\mathbb{Q}}[|D_s X_u|^{2p}] \, du\right)^{\frac{1}{2}} t^{p-\frac{1}{2}} \\
&\leq \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}} (E_{2p})^{\frac{1}{2}} t^p \leq \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}} (E_{2p})^{\frac{1}{2}} \qquad \text{(C.3.4)}
\end{aligned}$$

for some positive constant E_{2p} and

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[Y_t^p \left|\int_0^t \rho(u, X_u) D_s X_u \, dW_u\right|^p\right] \\
&\leq \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^t \rho(u, X_u) D_s X_u \, dW_u\right|^{2p}\right]\right)^{\frac{1}{2}} \qquad \text{(C.3.5)}
\end{aligned}$$

for $0 \leq t \leq 1$ and $s \geq 0$. Therefore, for $0 \leq t \leq 1$, $s \geq 0$, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|D_s Y_t|^p] \leq\ & 3^{p-1}\alpha^p \mathbb{E}^{\mathbb{Q}}[Y_t^p] + 3^{p-1}\alpha^{2p} \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}} (E_{2p})^{\frac{1}{2}} \\
& + 3^{p-1} \left(\mathbb{E}^{\mathbb{Q}}[|Y_t|^{2p}]\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^t \rho(u, X_u) D_s X_u \, dW_u\right|^{2p}\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

From Lemma 2.0.2, $\mathbb{E}^{\mathbb{Q}}[Y_t^p]$, $\mathbb{E}^{\mathbb{Q}}[Y_t^{2p}]$ are both bounded by some constant in $0 \leq t \leq 1$. Therefore, to establish the first inequality of Eq.(4.1.9), it is left to show that the expectation $\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^t \rho(u, X_u) D_s X_u \, dW_u\right|^{2p}\right]$ is also bounded from above by some constant in $0 \leq t \leq 1$.

Claim C.3.0.1. *For any fixed $s \geq 0$, a stochastic process $\left(\int_0^t \rho(u, X_u) D_s X_u \, dW_u\right)_{t \geq 0}$ is a continuous martingale adapted to the Brownian filtration $(\mathcal{F}_t^W)_{t \geq 0}$.*

Proof of Claim C.3.0.1. For any fixed $s \geq 0$, recall from Assumption 1 that a map $x \mapsto \sigma(s, x)$ is Borel measurable. Then, it is easy to check from

Eq.(C.3.1) that for any fixed $s \geq 0$, $\rho(u, X_u)D_sX_u$ is $\mathcal{F}_u^W$ measurable. From Eq.(C.3.1) and Assumption 1, check that

$$|D_sX_u|^p \leq \overline{\sigma}^p Y_u^p \Big[\max_{0\leq l\leq u}\left(X_l^p Y_l^{-p}\right)\Big]\mathbb{1}_{\{s\leq u\}}\,.$$

Note that $|\rho| \leq \alpha$ from Assumption 1. Therefore, from the Hölder inequality and Lemma 2.0.2, we can obtain

$$\begin{aligned}
\int_0^t \mathbb{E}^{\mathbb{Q}}[|\rho(u, X_u)D_sX_u|^2]\,du &\leq \alpha^2\int_0^t \mathbb{E}^{\mathbb{Q}}[|D_sX_u|^2]\,du \\
&\leq \alpha^2\overline{\sigma}^p\int_0^t \left(\mathbb{E}^{\mathbb{Q}}[Y_u^{2p}]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq l\leq u}\left(X_l^{2p}Y_l^{-2p}\right)\Big]\right)^{\frac{1}{2}}du \\
&\leq \alpha^2\overline{\sigma}^p\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq l\leq t}Y_l^{2p}\Big]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq l\leq t}\left(X_l^{2p}Y_l^{-2p}\right)\Big]\right)^{\frac{1}{2}}t \\
&< \infty
\end{aligned}$$

for every $t > 0$. Hence, the Ito integral $\int_0^t \rho(u, X_u)D_sX_u\,dW_u$ is well defined and it is a continuous martingale. □

According to Claim C.3.0.1, $\left(\int_0^t \rho(u, X_u)D_sX_u\,dW_u\right)_{t\geq 0}$ is a continuous martingale starting at 0. Thus, from the Burkholder–Davis–Gundy inequality, the Jensen inequality, Assumption 1, and Eq.(4.1.8),

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^t \rho(u, X_u)D_sX_u\,dW_u\right|^{2p}\right] &\leq C_p\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^t |\rho(u, X_u)|^2|D_sX_u|^2\,du\right)^p\right] \\
&\leq C_p\alpha^{2p}t^{p-1}\int_0^t \mathbb{E}^{\mathbb{Q}}[|D_sX_u|^{2p}]\,du \\
&\leq C_p\alpha^{2p}E_{2p}t^p \leq C_p\alpha^{2p}E_{2p} \qquad \text{(C.3.6)}
\end{aligned}$$

for $0 \leq t \leq 1$, $s \geq 0$ with some constant C_p, $E_{2p} > 0$. Hence, we complete the proof of Eq.(4.1.9).

Finally, we will examine the proof of Eq.(4.1.10). Note from Nualart (1995) that TD_sF can be expressed as $TD_sF = -\frac{1}{T}\int_0^T D_sY_t\,dt\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^{-2}$. Assume that $p \geq 1$. Observe from the Jensen inequality, the Hölder inequality, Lemma 2.0.2, and Eq.(4.1.9) that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|TD_sF|^p] &\leq \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T |D_sY_t|^p\,dt\right)\left(\max_{0\leq t\leq T} Y_t^{-2p}\right)\right] \\
&\leq \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|D_sY_t|^{2p}]\,dt\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq t\leq T} Y_t^{-4p}\Big]\right)^{\frac{1}{2}} \\
&\leq (E_{2p})^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\max_{0\leq u\leq 1} Y_u^{-4p}\Big]\right)^{\frac{1}{2}} < \infty
\end{aligned}
$$

for $0 \leq T \leq 1$ and any $s \geq 0$. This proves the first inequality of Eq.(4.1.10). The other two inequalities in Eq.(4.1.10) can be established similarly. □

C.4 Proof of Lemma 4.1.3

Proof. We will prove the first inequality of Eq.(4.1.11). Assume that $p \geq 1$. From Eqs.(C.3.2) and (C.3.3), we derive the following inequality under Assumption 1:

$$
\begin{aligned}
|D_sY_t - D_s\tilde{Y}_t| \leq\; & |Y_t\nu(s,X_s) - \tilde{Y}_t\nu(s,S_0)|\mathbb{1}_{\{s\leq t\}} \\
& + \alpha^2 Y_t \int_0^t |D_sX_u|\,du + Y_t\left|\int_0^t \rho(u,X_u)D_sX_u\,dW_u\right|
\end{aligned}
$$

for any $s \geq 0$. Therefore, from the inequality $(\frac{x+y+z}{3})^p \leq \frac{x^p+y^p+z^p}{3}$, $x,y,z > 0$ and the Jensen inequality,

$\mathbb{E}^{\mathbb{Q}}[|D_sY_t - D_s\tilde{Y}_t|^p]$

$$\leq 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|Y_t\nu(s,X_s)-\tilde{Y}_t\nu(s,S_0)\right|^p\right]\mathbb{1}_{\{s\leq t\}}$$
$$+\,3^{p-1}\alpha^{2p}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t|D_sX_u|^p\,du\right]t^{p-1}+3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t\rho(u,X_u)D_sX_u\,dW_u\right|^p\right].$$

Observe from the inequality Eq.(C.3.4) that we have already proved

$$\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\int_0^t|D_sX_u|^p\,du\right]t^{p-1}\leq F_pt^{\frac{p}{2}},$$

for some positive constant F_p in $0\leq t\leq 1$. Similarly, the inequalities Eq.(C.3.5), Eq.(C.3.6) imply the bound $\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\left|\int_0^t\rho(u,X_u)D_sX_u\,dW_u\right|^p\right]\leq F_pt^{\frac{p}{2}}$ in $0\leq t\leq 1$ for some positive constant F_p. Thus, it is sufficient to show that $\mathbb{E}^{\mathbb{Q}}\left[\left|Y_t\nu(s,X_s)-\tilde{Y}_t\nu(s,S_0)\right|^p\right]\mathbb{1}_{\{s\leq t\}}\leq F_pt^{\frac{p}{2}}$ for any $0\leq t\leq 1$, $s\geq 0$ with some positive constant F_p. Now, observe from the Jensen inequality and Assumption 1 that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|Y_t\nu(s,X_s)-\tilde{Y}_t\nu(s,S_0)\right|^p\right]\mathbb{1}_{\{s\leq t\}}$$
$$\leq 2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p|\nu(s,X_s)-\nu(s,S_0)|^p\right]\mathbb{1}_{\{s\leq t\}}+2^{p-1}\mathbb{E}^{\mathbb{Q}}\left[|\nu(s,S_0)|^p|Y_t-\tilde{Y}_t|^p\right]$$

$$\leq 2^{p-1}\alpha^p\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\max_{0\leq s\leq t}|X_s-S_0|^p\right]+2^{p-1}\alpha^p\mathbb{E}^{\mathbb{Q}}[|Y_t-\tilde{Y}_t|^p].$$

From Lemma 2.0.1 with $0\leq t\leq 1$, $\mathbb{E}^{\mathbb{Q}}[|Y_t-\tilde{Y}_t|^p]\leq B_pt^{\frac{p}{2}}$ for some constant $B_p>0$. Furthermore, by the Hölder inequality and the Doob L^p inequality,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[Y_t^p\max_{0\leq s\leq t}|X_s-S_0|^p\right]&\leq\left(\mathbb{E}^{\mathbb{Q}}[Y_t^{2p}]\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\left[\max_{0\leq s\leq t}|X_s-S_0|^{2p}\right]\right)^{\frac{1}{2}}\\&\leq\left(\mathbb{E}^{\mathbb{Q}}[Y_t^{2p}]\right)^{\frac{1}{2}}\left(\frac{2p}{2p-1}\right)^p\left(\mathbb{E}^{\mathbb{Q}}[|X_t-S_0|^{2p}]\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\mathbb{E}^{\mathbb{Q}}[Y_t^{2p}]$ is bounded by a constant in $0 \le t \le 1$ from Lemma 2.0.2, it is left to show that the expectation $\mathbb{E}^{\mathbb{Q}}[|X_t - S_0|^{2p}]$ is no greater than some constant F_p times t^p in $0 \le t \le 1$. This can be obtained from the inequality $(\frac{x+y+z}{3})^{2p} \le \frac{x^{2p}+y^{2p}+z^{2p}}{3}$, $x, y, z > 0$, and Lemma 2.0.1:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[|X_t - S_0|^{2p}] \\
&\le 3^{2p-1}\mathbb{E}^{\mathbb{Q}}[|X_t - \tilde{X}_t|^{2p}] + 3^{2p-1}\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_t - \hat{X}_t|^{2p}] + 3^{2p-1}\mathbb{E}^{\mathbb{Q}}[|\hat{X}_t - S_0|^{2p}] \\
&\le 3^{2p-1}B_{2p}t^{2p} + 3^{2p-1}B_{2p}t^{2p} + 3^{2p-1}\overline{\sigma}^{2p}S_0^{2p}M(2p)t^p\,,
\end{aligned}$$

where $M(p)$ denotes $\mathbb{E}^{\mathbb{Q}}[|Z|^p]$ while Z denotes a standard normal random variable. Note the last inequality comes from direct computation because $\hat{X}_t$ is a normal random variable. Hence, we prove the first inequality of Eq.(4.1.11).

Next, we prove the second inequality of Eq.(4.1.11). This easily comes from Eq.(C.3.2) and straightforward use of Assumption 1, the Jensen inequality, and Lemma 2.0.1. Assume that $p \le 1$ and observe that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[|D_s\tilde{Y}_t - D_s\hat{Y}_t|^p] &= \mathbb{E}^{\mathbb{Q}}[|\nu(s, S_0)(\tilde{Y}_t - 1)\mathbb{1}_{\{s \le t\}}|^p] \\
&\le \alpha^p\mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_t - 1|^p] \\
&\le 2^{p-1}\alpha^p\mathbb{E}^{\mathbb{Q}}[|\tilde{Y}_t - \hat{Y}_t|^p] + 2^{p-1}\alpha^p\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t - 1|^p] \\
&\le 2^{p-1}\alpha^pB_pt^p + 2^{p-1}\alpha^{2p}M(p)t^{\frac{p}{2}}\,,
\end{aligned}$$

where $M(p)$ denotes $\mathbb{E}^{\mathbb{Q}}[|Z|^p]$ while Z denotes a standard normal random variable. The last inequality comes from direct computation with regard to a normal random variable $\hat{Y}_t$. This completes the proof of Eq.(4.1.11).

Finally, we will prove Eq.(4.1.12). In Nualart (1995), Malliavin calculus gives

$$TD_sF = \frac{-\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^2}\,, \quad TD_s\tilde{F} = \frac{-\frac{1}{T}\int_0^T D_s\tilde{Y}_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2}\,.$$

If $p \geq 1$, by the Jensen inequality, $\mathbb{E}^{\mathbb{Q}}[|TD_sF - TD_s\tilde{F}|^p] \leq 2^{p-1}\mathbb{E}^{\mathbb{Q}}[L_T^p] + 2^{p-1}\mathbb{E}^{\mathbb{Q}}[R_T^p]$, where L_T, R_T are given by

$$L_T := \left| \frac{\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T Y_t\,dt\right)^2} - \frac{\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2} \right|,$$

$$R_T := \left| \frac{\frac{1}{T}\int_0^T D_sY_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2} - \frac{\frac{1}{T}\int_0^T D_s\tilde{Y}_t\,dt}{\left(\frac{1}{T}\int_0^T \tilde{Y}_t\,dt\right)^2} \right|.$$

Then, for A_T, which satisfies the equality

$$L_T = A_T \left| \frac{1}{T}\int_0^T Y_t\,dt - \frac{1}{T}\int_0^T \tilde{Y}_t\,dt \right|,$$

we observe from the Jensen inequality, the Hölder inequality, and Lemma 2.0.1 that

$$\mathbb{E}^{\mathbb{Q}}[L_T^p] \leq \left(\mathbb{E}^{\mathbb{Q}}[A_T^{2p}]\right)^{\frac{1}{2}} \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|Y_t - \tilde{Y}_t|^{2p}]\,dt\right)^{\frac{1}{2}} \leq \left(\mathbb{E}^{\mathbb{Q}}[A_T^{2p}]\right)^{\frac{1}{2}} \left(\frac{B_{2p}}{2p+1}\right)^{\frac{1}{2}} T^{\frac{p}{2}}$$

for $0 \leq T \leq 1$. From Lemmas 2.0.2 and 4.1.2, it is easy to prove that $\mathbb{E}^{\mathbb{Q}}[A_T^{2p}]$ is bounded above by some constant in $0 \leq T \leq 1$. Therefore, $\mathbb{E}^{\mathbb{Q}}[L_T^p] \leq F_pT^{\frac{p}{2}}$ for some positive constant F_p. Next, we examine $\mathbb{E}^{\mathbb{Q}}[R_T^p]$. From the Jensen inequality, the Hölder inequality, and Eq.(4.1.11) already established,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[R_T^p] &= \mathbb{E}^{\mathbb{Q}}\left[\left| \frac{1}{T}\int_0^T D_sY_t - D_s\tilde{Y}_t\,dt \right|^p (T\tilde{F})^{2p} \right] \\
&\leq \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|D_sY_t - D_s\tilde{Y}_t|^{2p}]\,dt\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}}[(T\tilde{F})^{4p}]\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \left(\frac{F_{2p}}{p+1}\right)^{\frac{1}{2}} T^{\frac{p}{2}} \left(\mathbb{E}^{\mathbb{Q}}[(T\tilde{F})^{4p}]\right)^{\frac{1}{2}} .$$

From Lemma 2.0.2, $\mathbb{E}^{\mathbb{Q}}[(T\tilde{F})^{4p}]$ is bounded above by a constant in $0 \leq T \leq 1$. This completes the proof. The second inequality of Eq.(4.1.12) can be obtained similarly except that we should additionally control the indicator function of $\{\frac{1}{T}\int_0^T \hat{Y}_t\, dt \geq \frac{1}{2}\}$ shown in the definition of $D_s^* \hat{F}$. However, we can resolve this subtle difference with the same technique as that used to prove Lemma 4.1.1 in Appendix C.2. □

C.5 Proof of Lemma 4.2.1

Proof. Observe that $(\hat{u}_s)_{s\geq 0}$ is adapted to the Brownian filtration $(\mathcal{F}_s^W)_{s\geq 0}$, and from Assumption 1,

$$\int_0^t \mathbb{E}^{\mathbb{Q}}[\hat{u}_s^2]\, ds \leq \int_0^t \frac{16\mathbb{E}^{\mathbb{Q}}[\hat{Y}_s^4]}{\underline{\sigma}^2 S_0^2}\, ds \leq \int_0^t \frac{16(1+6\alpha^2 s + 3\alpha^4 s^2)}{\underline{\sigma}^2 S_0^2}\, ds < \infty$$

for every $t > 0$. For the second inequality, we directly calculate the upper bound of $\mathbb{E}^{\mathbb{Q}}[\hat{Y}_s^4]$ from Assumption 1 and the fact that $\hat{Y}_s$ is a normal random variable. Thus, the Skorokhod integral $\delta(\hat{u})$ coincides with the Ito integral of $\hat{u}_s$ in $s \in [0,T]$. Furthermore, it is implied that $\left(\int_0^t \hat{u}_s\, dW_s\right)_{t\geq 0}$ is a continuous martingale starting at 0. Therefore, by the Burkholder–Davis–Gundy inequality and the Jensen inequality, if $p \geq 2$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{u})|^p] = \mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^T \hat{u}_s\, dW_s\right|^p\right] &\leq C_p \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T \hat{u}_s^2\, ds\right)^{\frac{p}{2}}\right] \\ &\leq C_p T^{\frac{p}{2}-1} \int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^p]\, ds\end{aligned}$$

for some constant $C_p > 0$. Note from Assumption 1 and the Doob L^p inequality that for $0 \leq s \leq T$,

$$\mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^p] \leq \frac{4^p}{\underline{\sigma}^p S_0^p} \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_s|^{2p}] \leq \frac{4^p}{\underline{\sigma}^p S_0^p} \mathbb{E}^{\mathbb{Q}}\left[\left(\max_{0 \leq s \leq T} |\hat{Y}_s|\right)^{2p}\right]$$
$$\leq \frac{4^p}{\underline{\sigma}^p S_0^p} \left(\frac{2p}{2p-1}\right)^{2p} \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_T|^{2p}].$$

This results in the following inequality.

$$\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^p] \, ds \leq \frac{4^p}{\underline{\sigma}^p S_0^p} \left(\frac{2p}{2p-1}\right)^{2p} \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_T|^{2p}] \, T.$$

Since $\hat{Y}_T$ is a normal random variable with mean 1, it is easy to confirm that $\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_T|^{2p}]$ is bounded by some constant, say $G_p > 0$, which depends only on p, for any $0 \leq T \leq 1$. From the inequalities established in this paragraph, we get that

$$\mathbb{E}^{\mathbb{Q}}[|\delta(\hat{u})|^p] \leq C_p \frac{4^p}{\underline{\sigma}^p S_0^p} \left(\frac{2p}{2p-1}\right)^{2p} G_p T^{\frac{p}{2}}$$

holds for any $0 \leq T \leq 1$. Rename the constant $C_p \frac{4^p}{\underline{\sigma}^p S_0^p} \left(\frac{2p}{2p-1}\right)^{2p} G_p$ as G_p to get the first inequality of Lemma 4.2.1.

Next, we will prove the second inequality of Eq.(4.2.2). For simplicity, we use the notation $g_s := \hat{u}_s - \frac{2}{\sigma(s, S_0) S_0}$ for the remainder of the proof. Note from the Burkholder–Davis–Gundy inequality and the Jensen inequality that if $p \geq 2$,

$$\mathbb{E}^{\mathbb{Q}}[|\delta(g)|^p] = \mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^T g_s \, dW_s\right|^p\right] \leq C_p \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T g_s^2 \, ds\right)^{\frac{p}{2}}\right]$$
$$\leq C_p T^{\frac{p}{2}-1} \int_0^T \mathbb{E}^{\mathbb{Q}}[|g_s|^p] \, ds. \tag{C.5.1}$$

Using the inequality $(\frac{x+y+z}{3})^p \le \frac{x^p+y^p+z^p}{3}$, $x, y, z > 0$, we get

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|g_s|^p] \le 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\hat{u}_s - \frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0}\mathbb{1}_{\{\hat{X}_s \ge \frac{S_0}{2}\}}\right|^p\right] \\ + 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0}\mathbb{1}_{\{\hat{X}_s < \frac{S_0}{2}\}}\right|^p\right] \\ + 3^{p-1}\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0} - \frac{2}{\sigma(s,S_0)S_0}\right|^p\right] .\end{aligned} \tag{C.5.2}$$

First, note that there is a positive constant $G_p^{(1)}$ that depends only on p such that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\left|\hat{u}_s - \frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0}\mathbb{1}_{\{\hat{X}_s \ge \frac{S_0}{2}\}}\right|^p\right] &\le \frac{4^p\alpha^p}{\underline{\sigma}^{2p}S_0^{2p}}\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_s|^{2p}|\hat{X}_s - S_0|^p] \\ &\le G_p^{(1)}s^{\frac{p}{2}}\end{aligned} \tag{C.5.3}$$

holds for any $0 \le s \le 1$. The first inequality comes from Assumption 1 . Since $\hat{Y}_s$, $\hat{X}_s$ are normal variables, it is easy to check that in $0 \le s \le 1$, $\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_s|^{2p}|\hat{X}_s - S_0|^p]$ is dominated by $s^{\frac{p}{2}}$ up to a constant multiplication. Therefore, we can choose a proper $G_p^{(1)}$. Second, observe that there is a constant $G_p^{(2)} > 0$ such that the following inequality holds for any $0 \le s \le 1$.

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0}\mathbb{1}_{\{\hat{X}_s < \frac{S_0}{2}\}}\right|^p\right] \le G_p^{(2)}s^{\frac{p}{2}} . \tag{C.5.4}$$

We obtain this bound from the following observations. For $0 \le s \le 1$, it is easy to check that $\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_s|^{2p}]$ is bounded by some constant. Furthermore, from the inequality $\mathbb{Q}\{\hat{X}_s < \frac{S_0}{2}\} \le N\left(-\frac{1}{2\overline{\sigma}\sqrt{s}}\right)$ established in (C.2.1), we can observe that $\mathbb{Q}\{\hat{X}_s < \frac{S_0}{2}\} = o(s^q)$ for any $q > 0$ as $s \to 0$. From these observations, we can find a suitable constant $G_p^{(2)}$. Third, observe from

Assumption 1 that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{2\hat{Y}_s^2}{\sigma(s,S_0)S_0}-\frac{2}{\sigma(s,S_0)S_0}\right|^p\right]\le\frac{2^p}{\underline{\sigma}^p S_0^p}\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_s^2-1|^{2p}]\le G_p^{(3)}s^{\frac{p}{2}}\quad\text{(C.5.5)}$$

holds for any $0\le s\le 1$, with some positive constant $G_p^{(3)}$. The constant $G_p^{(3)}$ can also be found by direct calculation with regard to a normal random variable $\hat{Y}_s$. Finally, from the inequalities Eqs.(C.5.1), (C.5.2), (C.5.3), (C.5.4), and (C.5.5) established in this paragraph, we obtain the desired result from

$$\mathbb{E}^{\mathbb{Q}}[|\delta(g)|^p]\le C_pT^{\frac{p}{2}-1}\int_0^T 3^{p-1}(G_p^{(1)}+G_p^{(2)}+G_p^{(3)})s^{\frac{p}{2}}\,ds\le G_pT^p\,,$$

with a suitable G_p. This completes the proof for Lemma 4.2.1. □

C.6 Proof for Proposition 4.2.1

Proof. First, we will prove Eq.(4.2.3). The proof comprises five claims: Claims C.6.0.1–C.6.0.5. Throughout the proof, we will use the notation $\Delta_A^*(T)$, which is defined as

$$\Delta_A^*(T):=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T\hat{X}_t\,dt\right)\frac{1}{T}\delta(\hat{u})\right].$$

Claim C.6.0.1.

$$\Delta_A^*(T)=\mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0+S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right]+\mathcal{O}(\sqrt{T}).$$

Proof of Claim C.6.0.1. From $\mathbb{E}^{\mathbb{Q}}[\delta(\hat{u})]=0$, $\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right]=0$, we get

$$\left|\Delta_A^*(T)-\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T\hat{X}_t\,dt\right)\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right]\right|$$

$$\leq \mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)-\Phi(S_0)\right|\frac{1}{T}|\delta(g)|\right],$$

where $g_s := \hat{u}_s - \frac{2}{\sigma(s,S_0)S_0}$. Use Assumption 3, the Jensen inequality, and the Hölder inequality to note that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)-\Phi(S_0)\right|\frac{1}{T}|\delta(g)|\right]\\
&\leq \beta\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t-S_0|^2]\,dt\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\delta(g)\right)^2\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

From the inequality $\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t-S_0|^2]\,dt = \mathcal{O}(T)$ and Lemma 4.2.1 where $\mathbb{E}^{\mathbb{Q}}[\delta(g)^2] \leq G_2T^2$, we get

$$\Delta_A^*(T) = \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right]+\mathcal{O}(\sqrt{T}).$$

Next, from the Fubini theorem with regard to a stochastic integral,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+\frac{1}{T}\int_0^T\int_0^t \sigma(s,S_0)S_0\,dW_s\,dt\right)\frac{1}{T}\int_0^T \frac{2}{\sigma(s,S_0)S_0}\,dW_s\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+\frac{S_0}{T}\int_0^T \sigma(s,S_0)(T-s)\,dW_s\right)\frac{1}{T}\int_0^T \frac{2}{\sigma(s,S_0)S_0}\,dW_s\right].
\end{aligned}$$

Then, $X := \int_0^T \sigma(s,S_0)(T-s)\,dW_s$ and $Y := \int_0^T \frac{2}{\sigma(s,S_0)S_0}\,dW_s$ are multivariate normal random variables satisfying

$$X \perp Y - \frac{T^2/S_0}{\int_0^T \sigma^2(s,S_0)(T-s)^2\,ds}X.$$

This is followed by

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+\frac{S_0}{T}X\right)\frac{1}{T}Y\right] \\
&=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_0+\frac{S_0}{T}X\right)\frac{1}{T}\left(\frac{T^2/S_0}{\int_0^T\sigma^2(s,S_0)(T-s)^2\,ds}X\right)\right] \\
&=\mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0+S_0\sigma_A(T)\sqrt{T}Z)}{S_0\sigma_A(T)\sqrt{T}}Z\right].
\end{aligned}
$$

Therefore, we get the desired result. □

Claim C.6.0.2.

$$
\Delta_A^*(T)=\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T\hat{X}_t\,dt\right)\frac{1}{T}\delta(\hat{u})\mathbb{1}_{\{\frac{1}{T}\int_0^T\hat{Y}_t\,dt\geq\frac{1}{2}\}}\right]+\mathcal{O}(\sqrt{T}).
$$

Proof of Claim C.6.0.2. We may assume that $\Phi(0)=0$. Otherwise, consider a translation $\Phi(\cdot)-\Phi(0)$ and follow the arguments below to get the result. Use Assumption 3, the Jensen inequality, and the Hölder inequality to get

$$
\begin{aligned}
&\left|\Delta_A^*(T)-\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T\hat{X}_t\,dt\right)\frac{1}{T}\delta(\hat{u})\mathbb{1}_{\{\frac{1}{T}\int_0^T\hat{Y}_t\,dt\geq\frac{1}{2}\}}\right]\right| \\
&=\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T\hat{X}_t\,dt\right)\frac{1}{T}\delta(\hat{u})\mathbb{1}_{\{\frac{1}{T}\int_0^T\hat{Y}_t\,dt<\frac{1}{2}\}}\right]\right| \\
&\leq\beta\left(\frac{1}{T}\int_0^T\mathbb{E}^{\mathbb{Q}}[|\hat{X}_t|^3]\,dt\right)^{\frac{1}{3}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\left(\frac{1}{\sqrt{T}}|\delta(\hat{u})|\right)^3\Big]\right)^{\frac{1}{3}}\left(\frac{\mathbb{Q}\left\{\frac{1}{T}\int_0^T\hat{Y}_t\,dt<\frac{1}{2}\right\}}{T\sqrt{T}}\right)^{\frac{1}{3}}.
\end{aligned}
$$

Since $\hat{X}_t$ is a normal random variable with mean S_0, we get the following asymptotic relation for small $T>0$: $\frac{1}{T}\int_0^T\mathbb{E}^{\mathbb{Q}}[|\hat{X}_t|^3]\,dt=\mathcal{O}(1)$. From

Lemma 4.2.1, we prove the bound $\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{\sqrt{T}}|\delta(\hat{u})|\right)^3\right] \leq G_3$. Therefore, from the inequality $\mathbb{Q}\left\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt < \frac{1}{2}\right\} \leq N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right)$ in Eq.(C.2.2), we get the desired result because $N\left(-\frac{\sqrt{3}}{2\alpha\sqrt{T}}\right) = o(T^q)$ for any $q > 0$ as $T \to 0$. □

Claim C.6.0.3.

$$\begin{aligned}\Delta_A^*(T) &- \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right]\\ &= \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T}).\end{aligned}$$

Proof of Claim C.6.0.3. Note the next equality from the definition of $\hat{F}$:

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\frac{1}{T}\delta(\hat{u})\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \geq \frac{1}{2}\}}\right] - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right]. \tag{C.6.1}$$

For simplicity, denote the expectation on the right-hand side of Eq.(C.6.1) by E_T only in the proof of this claim. Apply Claim C.6.0.2 to this equality. Then,

$$\Delta_A^*(T) - \mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\delta(\hat{u})\hat{F}\right] = E_T + \mathcal{O}(\sqrt{T}).$$

Thus, it suffices to show that

$$E_T = \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T}).$$

To show this, observe from the inequality $T\hat{F} \leq 2$, Assumption 3, the

Jensen inequality, and the Hölder inequality that

$$
\begin{aligned}
&\left| E_T - \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right]\right| \\
&\leq \frac{2}{\sqrt{T}}\,\mathbb{E}^{\mathbb{Q}}\left[\left|\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right) - \Phi(S_0)\right| \frac{1}{\sqrt{T}}|\delta(\hat{u})|\,\frac{1}{T}\int_0^T |\hat{Y}_t - 1|\,dt\right] \\
&\leq \frac{2\beta}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T |\hat{X}_t - S_0|\,dt\,\frac{1}{\sqrt{T}}|\delta(\hat{u})|\,\frac{1}{T}\int_0^T |\hat{Y}_t - 1|\,dt\right] \\
&\leq \frac{2\beta}{\sqrt{T}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t - S_0|^3]\,dt\right)^{\frac{1}{3}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\left(\frac{1}{\sqrt{T}}|\delta(\hat{u})|\right)^3\Big]\right)^{\frac{1}{3}} \\
&\times\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t - 1|^3]\,dt\right)^{\frac{1}{3}}.
\end{aligned}
$$

On the one hand, note from Lemma 4.2.1 that $\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{\sqrt{T}}|\delta(\hat{u})|\right)^3\right] \leq G_3$. On the other hand, by direct computation, $\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t - S_0|^3]\,dt$ and $\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t - 1|^3]\,dt$ are both $\mathcal{O}(T\sqrt{T})$ as $T \to 0$ because $\hat{X}_t$, $\hat{Y}_t$ are normal random variables with mean S_0, 1, respectively. This proves the claim. □

Claim C.6.0.4.

$$
\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T}).
$$

Proof of Claim C.6.0.4. This easily comes from straightforward use of the Jensen inequality and the Hölder inequality with Lemma 4.2.1. More precisely,

$$
\left|\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{u})\hat{F}\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right]\right|
$$

$$\leq \frac{1}{\sqrt{T}}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\sqrt{T}}|\delta(\hat{u})||T\hat{F}-1|\frac{1}{T}\int_0^T |\hat{Y}_t - 1|\,dt\right]$$
$$\leq \frac{1}{\sqrt{T}}\left(\mathbb{E}^{\mathbb{Q}}\Big[\left(\frac{1}{\sqrt{T}}|\delta(\hat{u})|\right)^3\Big]\right)^{\frac{1}{3}}\left(\mathbb{E}^{\mathbb{Q}}[|T\hat{F}-1|^3]\right)^{\frac{1}{3}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t-1|^3]\,dt\right)^{\frac{1}{3}}.$$

Here, $\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{\sqrt{T}}|\delta(\hat{u})|\right)^3\right] \leq G_3$ from Lemma 4.2.1. Furthermore, it is easy to check that $\mathbb{E}^{\mathbb{Q}}[|T\hat{F}-1|^3] = \mathcal{O}(T\sqrt{T})$ and $\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t - 1|^3]\,dt = \mathcal{O}(T\sqrt{T})$. This proves the claim. □

Claim C.6.0.5.

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] = \frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T}).$$

Proof of Claim C.6.0.5. Observe from the Jensen inequality and the Hölder inequality that

$$\left|\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right]\right|$$

$$\leq \frac{1}{T}\left(\mathbb{E}^{\mathbb{Q}}[\delta(g)^2]\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t-1|^2]\,dt\right)^{\frac{1}{2}},$$

where $g_s := \hat{u}_s - \frac{2}{\sigma(s,S_0)S_0}$. From Lemma 4.2.1, $\mathbb{E}^{\mathbb{Q}}[\delta(g)^2] \leq G_2 T^2$. Furthermore, by a direct computation, we can check that $\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{Y}_t - 1|^2]\,dt = \mathcal{O}(T)$. Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta(\hat{u})\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] + \mathcal{O}(\sqrt{T})\,.$$

Note from the Ito isometry that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt - 1\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T \frac{2}{\sigma(s,S_0)S_0}\,dW_s\right)\left(\frac{1}{T}\int_0^T \nu(s,S_0)(T-s)\,dW_s\right)\right] \\
&= \frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds\,.
\end{aligned}$$

This completes the proof. □

Concatenate the inequalities established through Claims C.6.0.1–C.6.0.5 to obtain Eq.(4.2.3).

Now, we will prove Eq.(4.2.4). We divide the proof into four claims.

Claim C.6.0.6.

$$\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] = \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof of Claim C.6.0.6. Observe from Assumption 3, the Jensen inequality, and the Hölder inequality that

$$\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\frac{1}{T}\int_0^T \hat{X}_t\,dt\right)\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] - \Phi(S_0)\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right]\right| \\
&\le \beta\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t - S_0|^2]\,dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[\hat{u}_s^2(TD_s^*\hat{F})^2]\,ds\right)^{\frac{1}{2}}.
\end{aligned}$$

From direct calculation, we obtain $\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{X}_t - S_0|^2]\,dt = \mathcal{O}(T)$. Note from Lemma 4.1.2 and the Hölder inequality that for any $0 \le s \le T \le 1$,

$$\mathbb{E}^{\mathbb{Q}}[\hat{u}_s^2(TD_s^*\hat{F})^2] \le \left(\mathbb{E}^{\mathbb{Q}}[\hat{u}_s^4]\right)^{\frac{1}{2}}(E_4)^{\frac{1}{2}} \le \frac{16}{\underline{\sigma}^2 S_0^2}\left(\mathbb{E}^{\mathbb{Q}}[\hat{Y}_s^8]\right)^{\frac{1}{2}}(E_4)^{\frac{1}{2}}.$$

We obtain the second inequality from the definition of $\hat{u}_s$ and Assumption 1. Moreover, it is easy to check that $s \mapsto \mathbb{E}^{\mathbb{Q}}[\hat{Y}_s^8]$ is bounded by some constant in $0 \le s \le 1$. Therefore, as $T \to 0$,

$$\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^2|TD_s^*\hat{F}|^2]\,ds = \mathcal{O}(1). \tag{C.6.2}$$

Thus, we achieve Claim C.6.0.6. □

Claim C.6.0.7.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right] + \mathcal{O}(\sqrt{T}).$$

Proof of Claim C.6.0.7. From the Jensen inequality and the Hölder inequality,

$$\begin{aligned}&\left|\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\right] - \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]\right|\\ &\le \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^2|TD_s^*\hat{F}|^2]\,ds\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathbb{Q}}\left[\left(\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2 - 1\right)^2\right]\right)^{\frac{1}{2}}.\end{aligned}$$

From the following two observations, we can complete the proof.

$$\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[|\hat{u}_s|^2|TD_s^*\hat{F}|^2]\,ds = \mathcal{O}(1), \quad \mathbb{E}^{\mathbb{Q}}\left[\left(\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2 - 1\right)^2\right] = \mathcal{O}(T).$$

The first equality has already been proved in Eq.(C.6.2) whereas the second equality comes easily from direct computation. □

Claim C.6.0.8.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right] + \mathcal{O}(\sqrt{T})\,.$$

Proof of Claim C.6.0.8. From the Jensen inequality and the Hölder inequality,

$$\Bigg|\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \hat{u}_s(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]$$
$$- \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]\Bigg|$$
$$\leq \left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[g_s^2(TD_s^*\hat{F})^2]\,ds\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[\hat{Y}_t^4]\,dt\right)^{\frac{1}{2}}\,,$$

where $g_s := \hat{u}_s - \frac{2}{\sigma(s,S_0)S_0}$. Note from Lemma 4.1.2 and the Hölder inequality that for any $0 \leq s \leq T \leq 1$,

$$\mathbb{E}^{\mathbb{Q}}[g_s^2(TD_s^*\hat{F})^2] \leq \left(\mathbb{E}^{\mathbb{Q}}[g_s^4]\right)^{\frac{1}{2}}(E_4)^{\frac{1}{2}}\,.$$

Observe from the inequalities Eqs.(C.5.2), (C.5.3), (C.5.4), and (C.5.5) that there exists some constant $G > 0$ such that $\mathbb{E}^{\mathbb{Q}}[g_s^4] \leq Gs^2$, for $0 \leq s \leq 1$. Furthermore, by direct computation, we can check that $\mathbb{E}^{\mathbb{Q}}[\hat{Y}_t^4]$ is bounded by some constant in small $T > 0$. This proves the claim. □

Claim C.6.0.9.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]$$
$$= -\frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds + \mathcal{O}(\sqrt{T})\,.$$

Proof of Claim C.6.0.9. From the definition of $D_s^*\hat{F}$ and the computation of $D_s\hat{Y}_t$ in Eq.(C.3.2), we get

$$D_s^*\hat{F} = -\frac{\int_0^T D_s\hat{Y}_t\,dt}{\left(\int_0^T \hat{Y}_t\,dt\right)^2}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \geq \frac{1}{2}\}} = -\frac{1}{T^2}\frac{\nu(s,S_0)(T-s)}{\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2}\mathbb{1}_{\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \geq \frac{1}{2}\}}\,.$$

Using this identity,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{2}{\sigma(s,S_0)S_0}(D_s^*\hat{F})\,ds\left(\frac{1}{T}\int_0^T \hat{Y}_t\,dt\right)^2\right]$$
$$= -\frac{2}{S_0}\frac{1}{T^2}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}(T-s)\,ds\,\mathbb{Q}\left\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt \geq \frac{1}{2}\right\}\,.$$

From Eq.(C.2.2), $\mathbb{Q}\left\{\frac{1}{T}\int_0^T \hat{Y}_t\,dt < \frac{1}{2}\right\} = o(T^q)$ for any $q > 0$ as $T \to 0$. Hence, we prove the claim. $\square$

Combining Claims C.6.0.6–C.6.0.9, we finally get Eq.(4.2.4). Here, we complete the proof of Proposition 4.2.1. $\square$

C.7 Proof for Proposition 4.3.3

Proof. Here, we only sketch the proof. A rigorous proof can be presented by routine use of well-known inequalities such as the Jensen inequality and the Hölder inequality, which have already been used throughout Appendix C.6. Hence, the details will be omitted.

First, we will study Eq.(4.3.5). Observe that

$$
\begin{aligned}
&\frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\delta(\hat{h})\hat{G}\right] - \frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_0)\delta(\hat{h})\hat{G}\right] \\
&= \frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\left(\Phi(\hat{X}_T)-\Phi(S_0)\right)\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)\hat{G}\right] + \mathcal{O}(\sqrt{T}) \quad \text{(C.7.1)}\\
&= \frac{1}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\left(\Phi(\hat{X}_T)-\Phi(S_0)\right)\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)S_0\right] + \mathcal{O}(\sqrt{T}) \quad \text{(C.7.2)}\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_0+S_0\sigma_E(T)\sqrt{T}Z)}{S_0\sigma_E(T)\sqrt{T}}Z\right] + \mathcal{O}(\sqrt{T})\,. \quad \text{(C.7.3)}
\end{aligned}
$$

The first equality Eq.(C.7.1) comes from $\mathbb{E}^{\mathbb{Q}}[|\hat{X}_T - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$, Lemma 4.3.2, and $\mathbb{E}^{\mathbb{Q}}[\hat{G}^p] = \mathcal{O}(1)$ for any $p > 0$. The second inequality Eq.(C.7.2) holds because $\mathbb{E}^{\mathbb{Q}}[|\hat{G} - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$ and $\mathbb{E}^{\mathbb{Q}}\left[\left|\delta\left(\frac{2}{\sigma(\cdot,S_0)S_0}\right)\right|^p\right] = \mathcal{O}(T^{\frac{p}{2}})$ for any $p > 0$. We obtain the last equality Eq.(C.7.3) from direct manipulation with regard to a multivariate normal variable. The same type of manipulation technique has already been used to prove Claim C.6.0.1 in Appendix C.6. Similarly, we get

$$
\begin{aligned}
&\frac{\Phi(S_0)}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{h})\hat{G}\right] \\
&= \frac{\Phi(S_0)}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\delta(\hat{h})(\hat{G}-S_0)\right] \quad \text{(C.7.4)}\\
&= \frac{\Phi(S_0)}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)(\hat{G}-S_0)\right] + \mathcal{O}(\sqrt{T}) \quad \text{(C.7.5)}\\
&= \frac{\Phi(S_0)}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)(\hat{G}-S_0)\hat{Y}_T\right] + \mathcal{O}(\sqrt{T}) \quad \text{(C.7.6)}\\
&= \frac{\Phi(S_0)}{S_0T}\mathbb{E}^{\mathbb{Q}}\left[\delta\left(\frac{1}{\sigma(\cdot,S_0)S_0}\right)(\hat{X}_T-S_0\hat{Y}_T)\right] + \mathcal{O}(\sqrt{T}) \quad \text{(C.7.7)}\\
&= \frac{\Phi(S_0)}{S_0T}\int_0^T \frac{\sigma(s,S_0)-\nu(s,S_0)}{\sigma(s,S_0)}\,ds + \mathcal{O}(\sqrt{T})\,. \quad \text{(C.7.8)}
\end{aligned}
$$

For the first equality Eq.(C.7.4), we use $\mathbb{E}^{\mathbb{Q}}[\delta(\hat{h})] = 0$. Lemma 4.3.2 and $\mathbb{E}^{\mathbb{Q}}[|\hat{G} - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$ for any $p > 0$ are used to establish the second equality Eq.(C.7.5). Next, the equality Eq.(C.7.6) is easily obtained from $\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_T - 1|^p] = \mathcal{O}(T^{\frac{p}{2}})$. The fourth equality Eq.(C.7.7) follows from the observation $\mathbb{Q}\{\hat{Y}_T < \frac{1}{2}\} = o(T^q)$ for any $q > 0$. This is similar to Eqs.(C.2.1) and (C.2.2), which are used to analyze the Asian option. The last equality Eq.(C.7.8) comes from the Ito isometry. From these arguments, we can prove Eq.(4.3.5).

Now, we analyze Eq.(4.3.6) from a series of asymptotic relations below.

$$
\begin{aligned}
&\frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \hat{h}_s\hat{H}_s\,ds\right] \\
&= \frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \frac{1}{\sigma(s,S_0)S_0}\hat{H}_s\,ds\right] + \mathcal{O}(\sqrt{T}) && \text{(C.7.9)}\\
&= \frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \frac{1}{\sigma(s,S_0)S_0}\hat{H}_s\,ds\,\hat{Y}_T^2\right] + \mathcal{O}(\sqrt{T}) && \text{(C.7.10)}\\
&= \frac{1}{T}\,\mathbb{E}^{\mathbb{Q}}\left[\Phi(\hat{X}_T)\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)S_0}\,ds\,\hat{X}_T\right] + \mathcal{O}(\sqrt{T}) && \text{(C.7.11)}\\
&= \frac{\Phi(S_0)}{T}\int_0^T \frac{\nu(s,S_0)}{\sigma(s,S_0)}\,ds + \mathcal{O}(\sqrt{T})\,. && \text{(C.7.12)}
\end{aligned}
$$

Note that the first equality Eq.(C.7.9) is obtained by the asymptotic relations $\mathbb{E}^{\mathbb{Q}}[|\hat{X}_T|^p] = \mathcal{O}(1)$, $\sup_{s\geq 0}\mathbb{E}^{\mathbb{Q}}[|\hat{H}_s|^p] = \mathcal{O}(1)$ and $\mathbb{E}^{\mathbb{Q}}\left[\left|\hat{h}_s - \frac{1}{\sigma(s,S_0)S_0}\right|^p\right] = \mathcal{O}(s^{\frac{p}{2}})$, as $T \to 0$. Similarly, we get the second equality Eq.(C.7.10) using $\mathbb{E}^{\mathbb{Q}}[|\hat{Y}_T^2 - 1|^p] = \mathcal{O}(T^{\frac{p}{2}})$. The third equality Eq.(C.7.11) comes directly from the definition of $\hat{H}_s$ with the observation that $\mathbb{Q}\{\hat{Y}_T < \frac{1}{2}\} = o(T^q)$, for any $q > 0$. The last equality Eq.(C.7.12) follows from $\mathbb{E}^{\mathbb{Q}}[|\hat{X}_T - S_0|^p] = \mathcal{O}(T^{\frac{p}{2}})$, $p > 0$. The proof for Eq.(4.3.6) is thus complete. $\square$

Appendix D

Detailed proof of Chapter 7

D.1 Proof for Theorem 7.2.1

Proof. First, we will show Eq.(7.2.1). Using Lemma C.1.1, the Hölder inequality, and the Jensen inequality, for $1/p + 1/p^{'} = 1\,,\ 1 < p\,, p^{'} < \infty\,,$

$$
\begin{aligned}
\Delta_A^{\text{call}}(T) &= \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_t\,dt\mathbb{1}_{\{\frac{1}{T}\int_0^T S_t\,dt \geq K\}}\right] \\
&\leq \frac{e^{-rT}}{S_0}\left(\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}}[S_t^p]\,dt\right)^{\frac{1}{p}}\left(\mathbb{Q}\left\{\frac{1}{T}\int_0^T S_t\,dt \geq K\right\}\right)^{\frac{1}{p^{'}}} \\
&\leq \frac{e^{-rT}}{S_0}\left(\frac{1}{T}\int_0^T S_0^p e^{p(r-q)t}e^{K(p)\overline{\sigma}^2 t}\,dt\right)^{\frac{1}{p}}\left(\mathbb{Q}\left\{\frac{1}{T}\int_0^T S_t\,dt \geq K\right\}\right)^{\frac{1}{p^{'}}}\,.
\end{aligned}
$$

For the last inequality, we use Eq.(A.2.1) with $K(p) := \frac{p(p-1)}{2} \vee 0\,.$ By taking $T \to 0\,,$ we get the following inequality from Remark 7.1.1,

$$
\limsup_{T\to 0} T\log\Delta_A^{\text{call}}(T) \leq \frac{-\mathcal{I}(K, S_0)}{p^{'}}\,.
$$

Take $p^{'} \to 1$ to get an upper bound. Next, a lower bound for Eq.(7.2.1) is obtained from the following inequality and Remark 7.1.1,

$$\Delta_A^{\text{call}}(T) = \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_t\,dt\,\mathbb{1}_{\{\frac{1}{T}\int_0^T S_t\,dt \geq K\}}\right] \geq \frac{e^{-rT}K}{S_0}\mathbb{Q}\left\{\frac{1}{T}\int_0^T S_t\,dt \geq K\right\}.$$

Proving Eq.(7.2.2) is simpler. From Lemma C.1.1, an upper bound for Eq.(7.2.2) is easily obtained from

$$-\Delta_A^{\text{put}}(T) = \frac{e^{-rT}}{S_0}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_t\,dt\,\mathbb{1}_{\{K\geq\frac{1}{T}\int_0^T S_t\,dt\}}\right] \leq \frac{e^{-rT}K}{S_0}\mathbb{Q}\left\{K \geq \frac{1}{T}\int_0^T S_t\,dt\right\}.$$

For a lower bound, observe that for any $0 < L < K$,

$$\begin{aligned}
-\Delta_A^{\text{put}}(T) &= \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_t\,dt\,\mathbb{1}_{\{K\geq\frac{1}{T}\int_0^T S_t\,dt\}}\right]\\
&\geq \frac{e^{-rT}}{S_0}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_t\,dt\,\mathbb{1}_{\{K\geq\frac{1}{T}\int_0^T S_t\,dt>L\}}\right]\\
&\geq \frac{e^{-rT}\epsilon}{S_0}\mathbb{Q}\left\{K \geq \frac{1}{T}\int_0^T S_t\,dt > L\right\}.
\end{aligned}$$

Therefore, from Remark 7.1.1,

$$\begin{aligned}
\liminf_{T\to 0} T\log\left(-\Delta_A^{\text{put}}(T)\right) &\geq \liminf_{T\to 0} T\log\left(\mathbb{Q}\left\{K \geq \frac{1}{T}\int_0^T S_t\,dt > L\right\}\right)\\
&= -\inf_{K\geq x>L}\mathcal{I}(x, S_0).
\end{aligned}$$

The last term is equal to $-\mathcal{I}(K, S_0)$ because of Proposition 7.1.1. $\square$

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국문초록

이 논문에서는 아시안 옵션의 가격과 헷징 포트폴리오의 단기적 행태에 대해 이야기한다. 우리는 지역 변동성 모델 아래, 횔더 연속인 수익함수를 가지는 아시안 옵션의 위험중립적 가격과 델타 가치를 고려한다. 이 논문의 핵심 아이디어는 지역 변동성 모델이 짧은 만기 T에서 가우시안 과정으로 근사될 수 있다는 점에 있다. 이 근사 이론과 말리아빈 미적분을 결합하여, 우리는 단기 아시안 옵션의 가격과 델타 가치는 아래 식의 변동성을 가지는 단기 유러피안 옵션의 가격과 델타 가치로 각각 근사됨을 결론 내린다.

$$\sigma_A(T) := \sqrt{\frac{1}{T^3}\int_0^T \sigma^2(t,S_0)(T-t)^2\,dt}\,,$$

여기서 $\sigma(\cdot,\cdot)$ 는 지역 변동성 함수를 의미하고 S_0 는 초기 주식 가격을 의미한다. 또한, 우리는 근사값의 수렴비율이 수익함수의 횔더 상수에 의존함을 증명한다. 마지막으로, 큰 편차 이론에 기반하여 단기 콜옵션과 풋옵션의 근사값에 대해서 논의해본다.

주요어휘: 아시안 옵션, 단기, 횔더 연속, 지역 변동성 모델, 가우시안 과정, 말리아빈 미적분, 큰 편차 이론
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